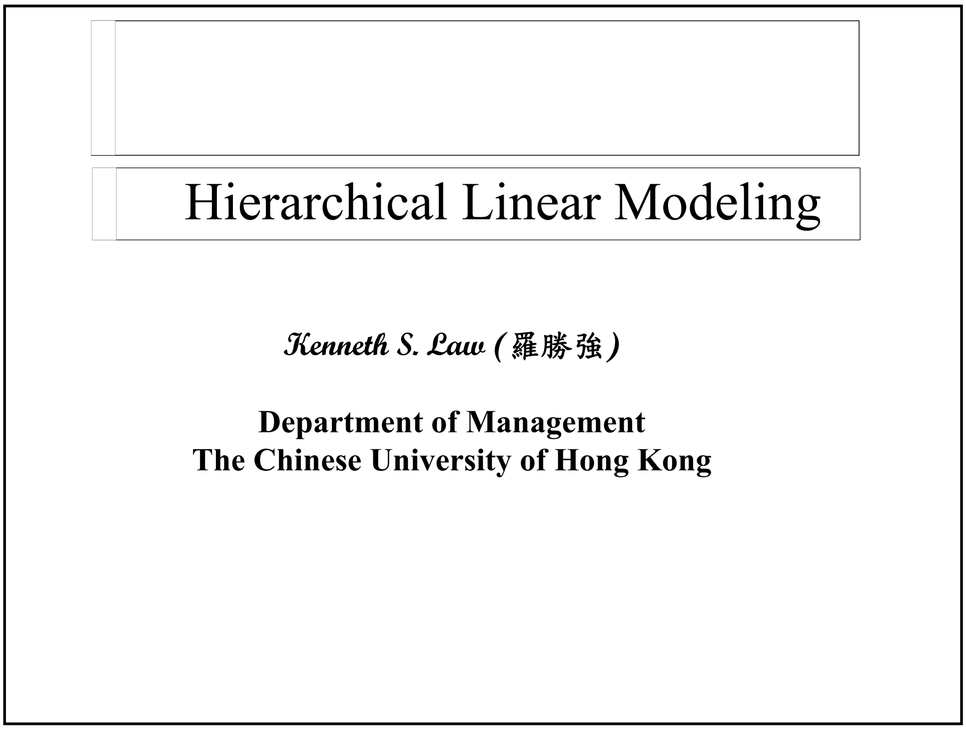
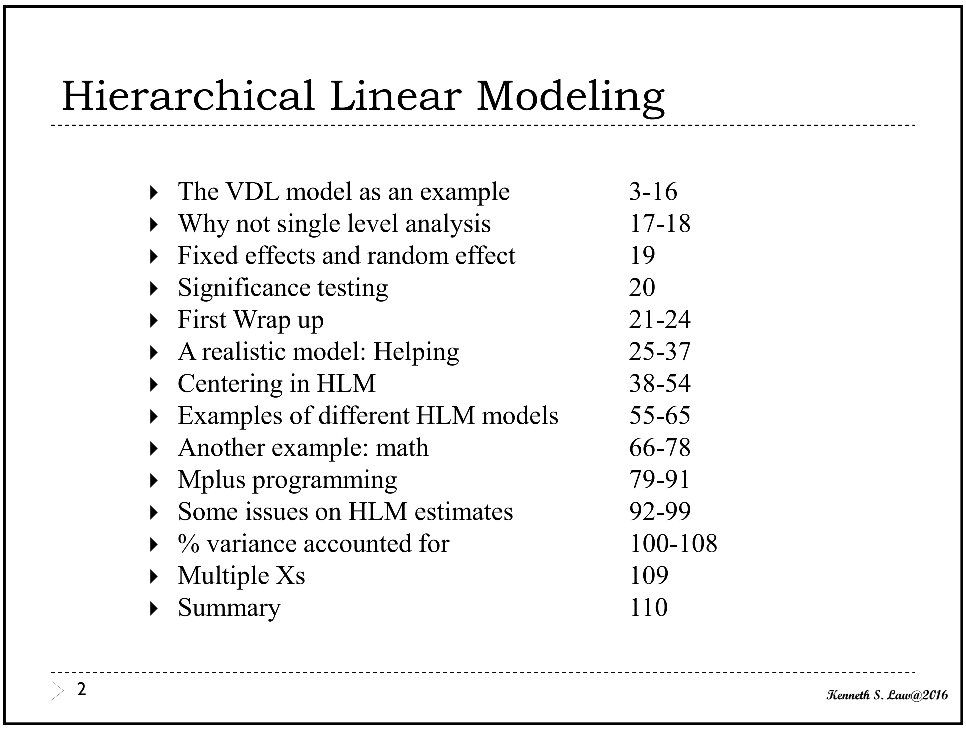


Hierarchical Linear Modeling

Kenneth S. Law (羅勝強)

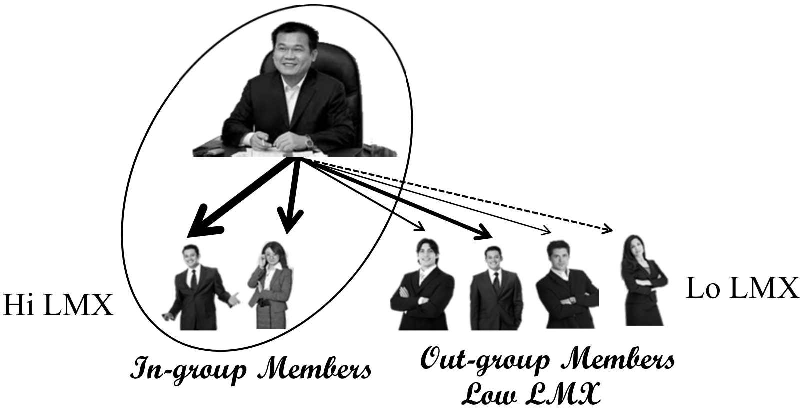
Department of Management
The Chinese University of Hong Kong



Hierarchical Linear Modeling

The Vertical Dyadic Linkage Model

The VDL model says that supervisors deliberately develop different exchange relationship with different subordinates.



Hi LMX

Lo LMX

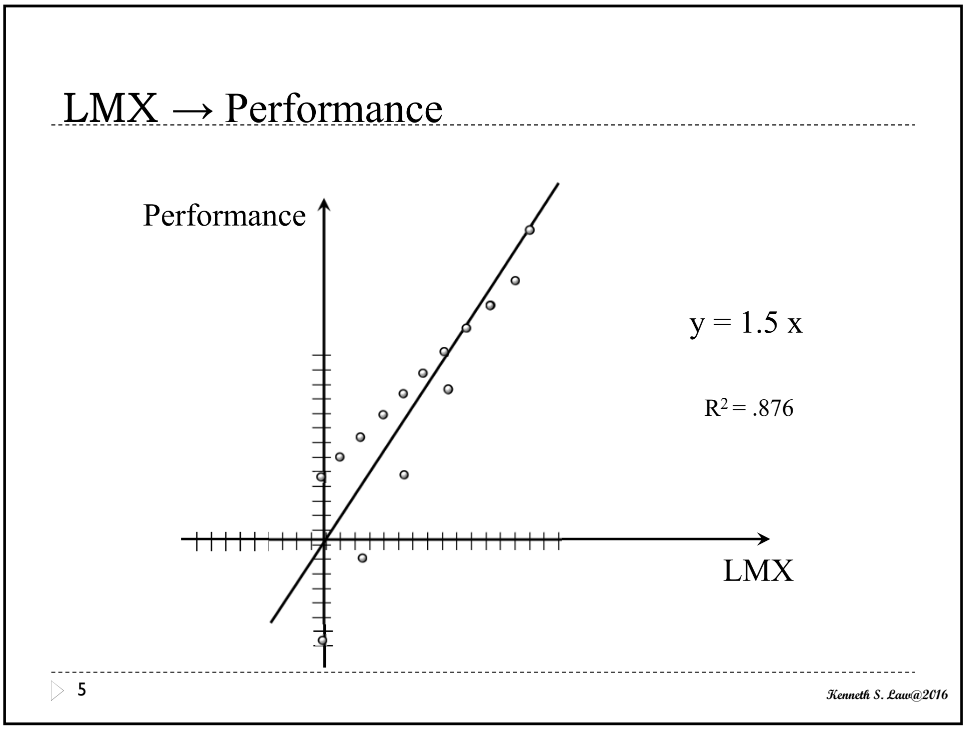
In-group Members

Out-group Members
Low LMX

HLM

Leader Member Exchange

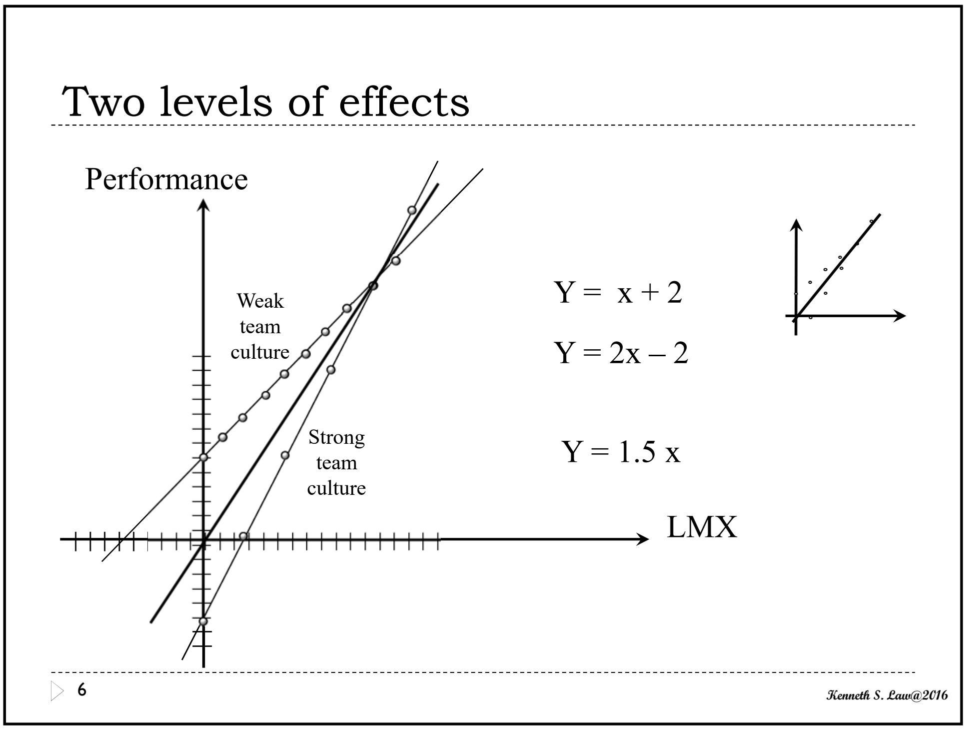
Subordinate Performance



LMX → Performance

y = 1.5 x

$R^2 = .876$



Two levels of effects

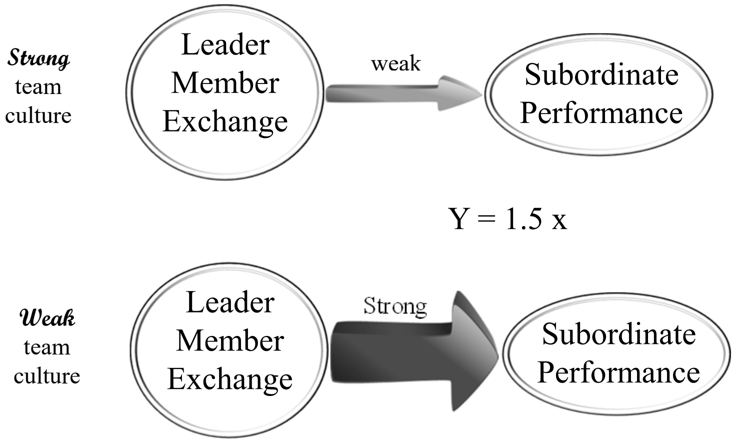
Y = x + 2

Y = 2x – 2

Y = 1.5 x

HLM

Y = x + 2

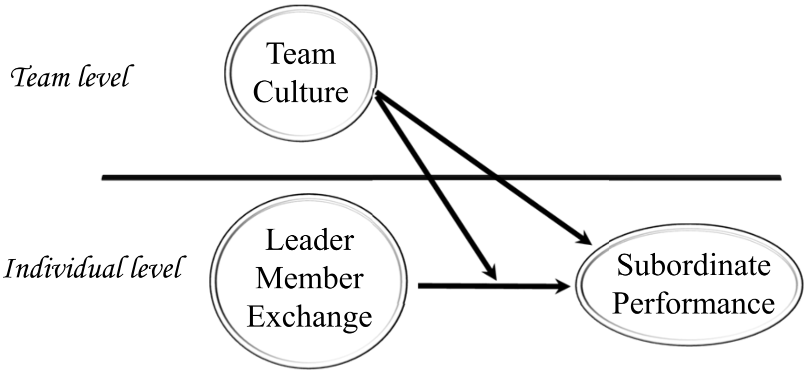


Y = 1.5 x

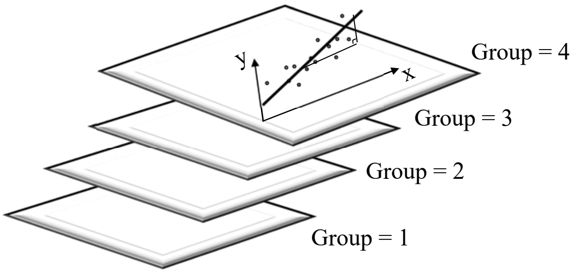
Y = 2x – 2

HLM

An example of the effects of team culture and LMX on performance of subordinates.

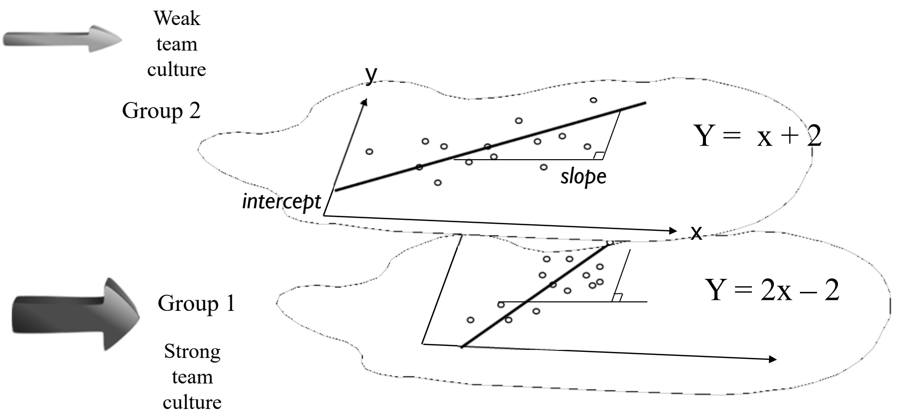


Figurative Illustration of HLM



x = LMX
y = performance

A simple case of two groups



The purpose of HLM

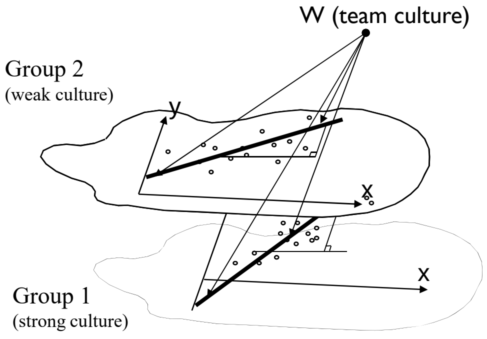
Group	Culture	Intercept	Slope	Equation
Group 2	Weak culture	-2	2	Y = 2x – 2
Group 1	Strong culture	+2	1	Y = x + 2

- We need to estimate the intercept and slope of each group separately.
  y = intercept + slope x + error
- What happened after estimating the intercept and slope of each group?
- We try to find a group-level variable to predict the variable intercept and slope in each group.

$$Intercept = \gamma_{00} + \gamma_{01}W + u_1$$

$$slope = \gamma_{10} + \gamma_{11}W + u_2$$

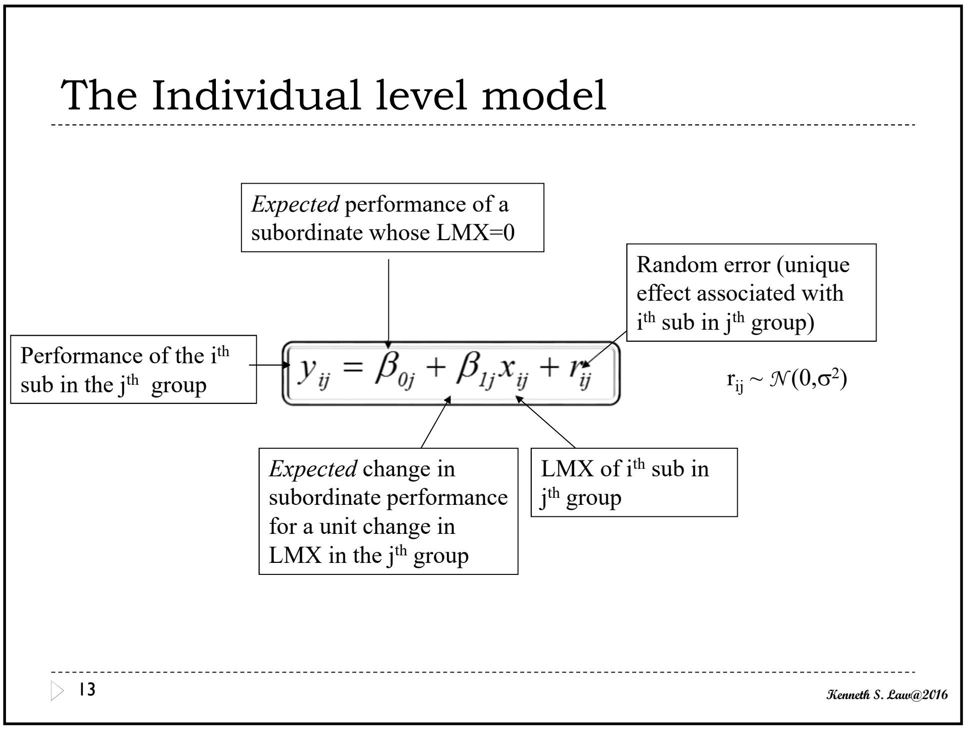
Cross level analysis



1. Within each level, there is a regression analysis;

- Level 1
  $y = \beta_{01} + \beta_{11}x + \varepsilon_1$
- Level 2
  $y = \beta_{02} + \beta_{12}x + \varepsilon_2$

2. The differential effect of x on y, (i.e., the intercept β_0 and slope β_1) at each level can be predicted by a group level variable (W).



The Individual level model

Expected performance of a subordinate whose LMX=0

Random error (unique effect associated with ith sub in jth group)

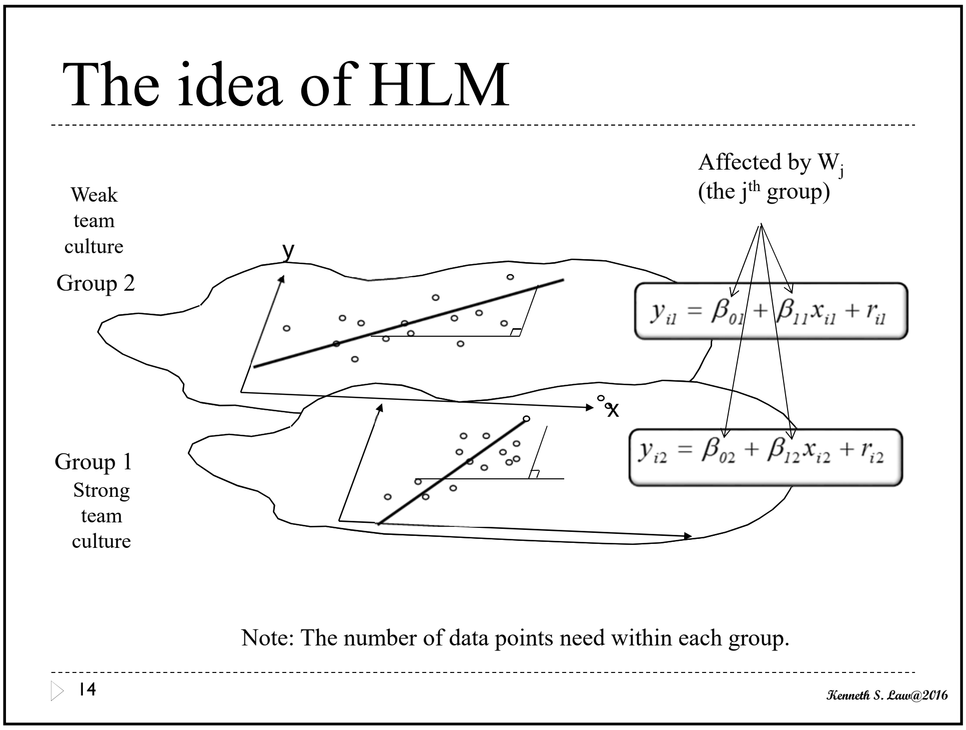
Performance of the ith sub in the jth group

$$\boxed{y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij}}$$

$r_{ij} \sim \mathcal{N}(0,\sigma^2)$

Expected change in subordinate performance for a unit change in LMX in the jth group

LMX of ith sub in jth group



The idea of HLM

Affected by W_j (the jth group)

Weak team culture

Group 2

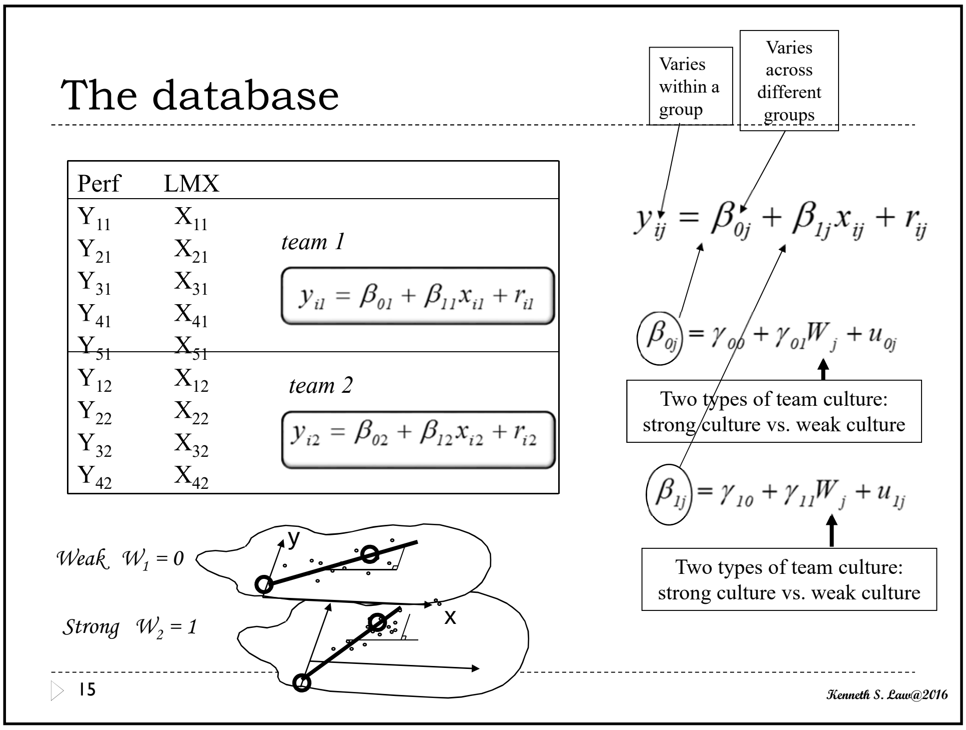
$$\boxed{y_{i1} = \beta_{01} + \beta_{11}x_{i1} + r_{i1}}$$

$$\boxed{y_{i2} = \beta_{02} + \beta_{12}x_{i2} + r_{i2}}$$

Group 1

Strong team culture

Note: The number of data points need within each group.



The database

Varies within a group

Varies across different groups

Perf	LMX	
Y_{11}	X_{11}	
Y_{21}	X_{21}	*team 1*
Y_{31}	X_{31}	$y_{i1} = \beta_{01} + \beta_{11}x_{i1} + r_{i1}$
Y_{41}	X_{41}	
Y_{51}	X_{51}	
Y_{12}	X_{12}	*team 2*
Y_{22}	X_{22}	$y_{i2} = \beta_{02} + \beta_{12}x_{i2} + r_{i2}$
Y_{32}	X_{32}	
Y_{42}	X_{42}	

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij}$$

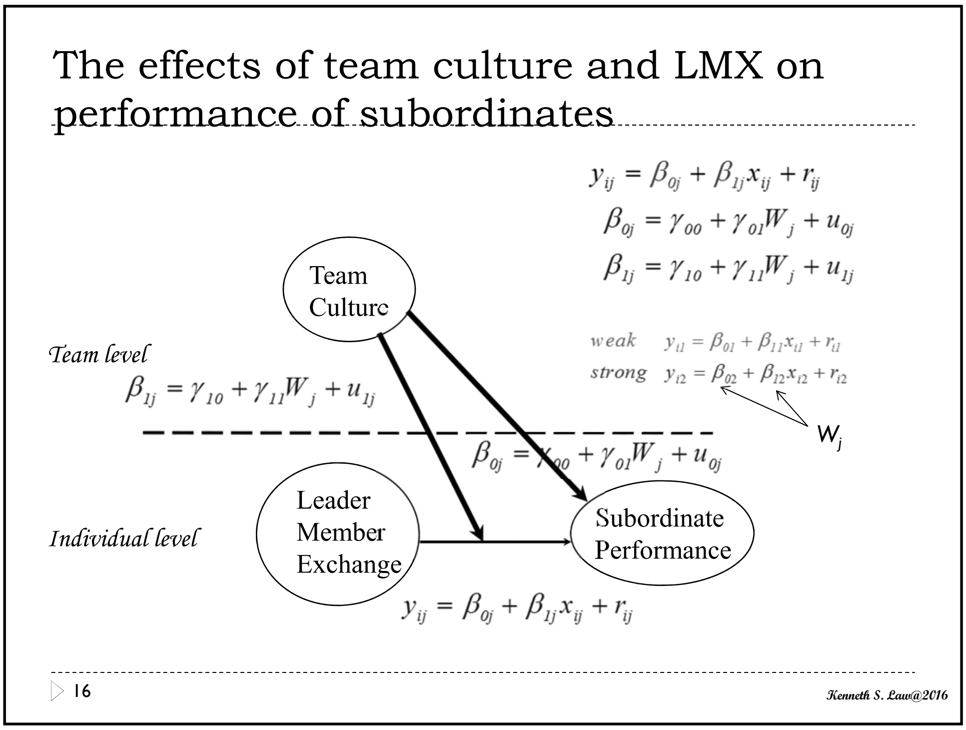
$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$

Two types of team culture: strong culture vs. weak culture

$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$$

Two types of team culture: strong culture vs. weak culture

Kenneth S. Law@2016



The effects of team culture and LMX on performance of subordinates

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$$

Team level

$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$

Individual level

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij}$$

weak $y_{i1} = \beta_{01} + \beta_{11}x_{i1} + r_{i1}$

strong $y_{i2} = \beta_{02} + \beta_{12}x_{i2} + r_{i2}$

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Why need two levels?

Y	LMX	Culture	
Y_{11}	X_{11}	0	$y_{i1} = \beta_{01} + \beta_{11}x_{i1} + r_{i1}$
Y_{21}	X_{21}	0	
Y_{31}	X_{31}	0	
Y_{41}	X_{41}	0	
Y_{51}	X_{51}	0	
Y_{12}	X_{12}	1	$y_{i2} = \beta_{02} + \beta_{12}x_{i2} + r_{i2}$
Y_{22}	X_{22}	1	
Y_{32}	X_{32}	1	
Y_{42}	X_{42}	1	

$$\boxed{y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij}}$$

$$\boxed{\begin{aligned}\beta_{0j} &= \gamma_{00} + \gamma_{01}W_j + u_{0j}\\ \beta_{1j} &= \gamma_{10} + \gamma_{11}W_j + u_{1j}\end{aligned}}$$

Why can't we use multiple regression and simply specify the following model?

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + \beta_{2j}w_j + r_{ij}$$

Correlated Error Terms

Level 1 $$\boxed{y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij}}$$

$$\boxed{y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + \beta_{2j}w_j + r_{ij}}$$

Level 2

$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$

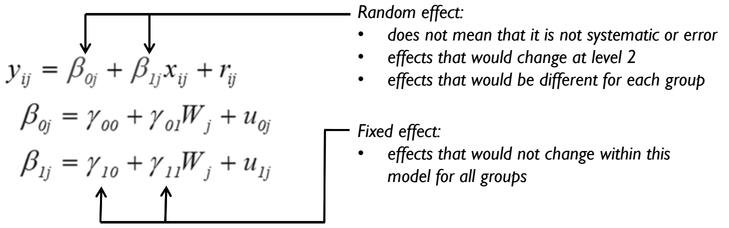
$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$$

$$\boxed{y_{ij} = \left[\gamma_{00} + \gamma_{01}w_j + u_{0j}\right] + \left[\gamma_{10} + \gamma_{11}w_j + u_{1j}\right]x_{ij} + r_{ij}}$$

$$y_{ij} = \gamma_{00} + \gamma_{10}x_{ij} + \gamma_{01}w_j + \gamma_{11}x_{ij}w_j + \left[r_{ij} + u_{0j} + x_{ij}u_{1j}\right]$$

$$y_{ij} = b_{0j} + b_{1j}x_{ij} + b_{2j}w_j + b_{3j}x_{ij}w_j + r_{ij}$$

Fixed effect vs. Random effect



$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$

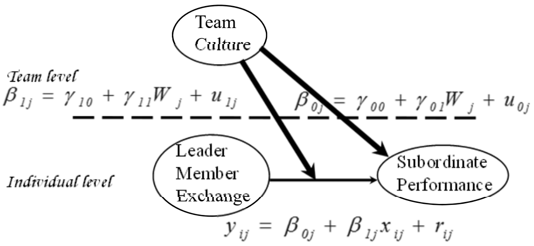
$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$$

Random effect:

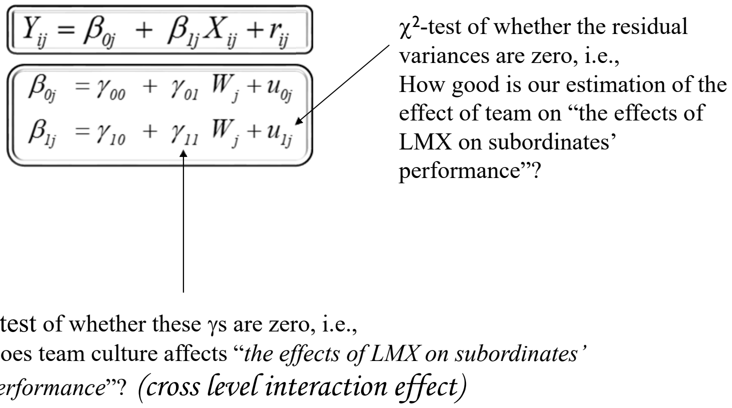
- *does not mean that it is not systematic or error*
- *effects that would change at level 2*
- *effects that would be different for each group*

Fixed effect:

- *effects that would not change within this model for all groups*



Significance Tests



$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij}$$

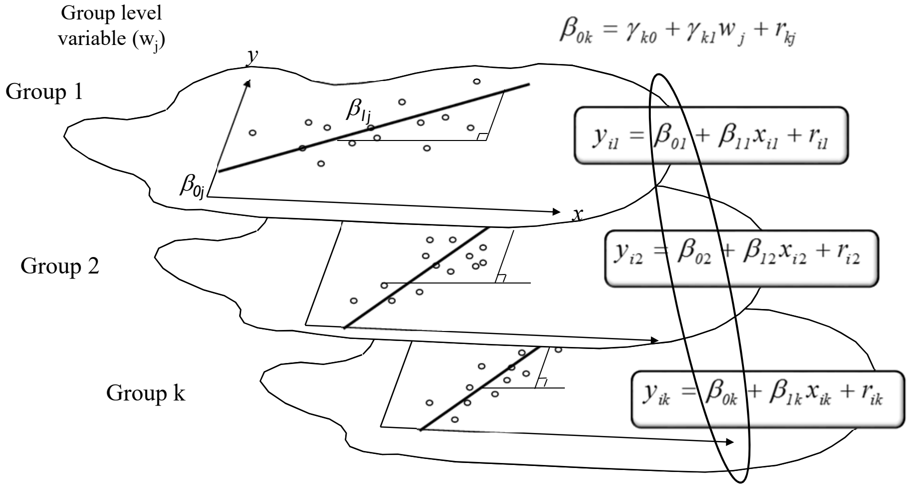
$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$$

χ^2-test of whether the residual variances are zero, i.e.,
How good is our estimation of the effect of team on "the effects of LMX on subordinates' performance"?

t-test of whether these γs are zero, i.e.,
Does team culture affects "*the effects of LMX on subordinates' performance*"? *(cross level interaction effect)*

Wrap up: The idea of HLM



A simple Two-Level Model

Level 1 (e.g., subs) $y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + r_{ij}$

Random Effects

Level 2 (e.g., teams) $\beta_{0j} = \gamma_{00} + \gamma_{01} w_j + u_{0j}$

$\beta_{1j} = \gamma_{10} + \gamma_{11} w_j + u_{1j}$

Fixed Effects

Where we assume:

$E(r_{ij}) = 0$ $\quad$ $Var(r_{ij}) = \sigma^2$

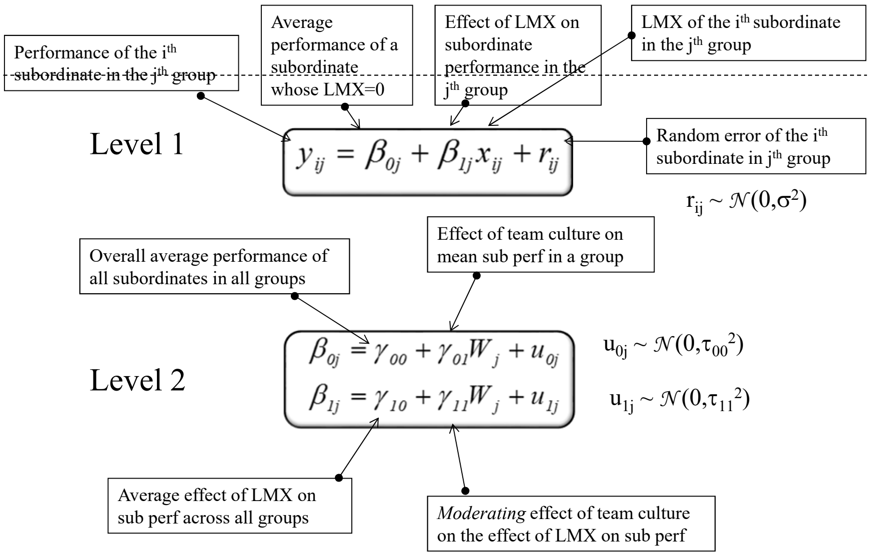
$Cov(u_{0j}, r_{ij}) = Cov(u_{1j}, r_{ij}) = 0$

$$E\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad Var\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} = \begin{bmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{bmatrix} = \mathbf{T}$$

Variance-Covariance Matrix of the residuals

In multi-level analyses, a quantity is random means that it fluctuates over units in the population. Therefore, estimates that are constant across levels is consider as fixed effects.

Sourcese: Snijders, T.A.B. "Fixed and Random Effects" In B.S. Everitt and D.C. Howell (eds), Encyclopedia of Statistics in Behavioral Science. Vol. 2, 664-665. Chicester (etc.):Wiley, 2005.



Performance of the ith subordinate in the jth group

Average performance of a subordinate whose LMX=0

Effect of LMX on subordinate performance in the jth group

LMX of the ith subordinate in the jth group

Level 1

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + r_{ij}$$

Random error of the ith subordinate in jth group

$r_{ij} \sim \mathcal{N}(0,\sigma^2)$

Overall average performance of all subordinates in all groups

Effect of team culture on mean sub perf in a group

Level 2

$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$$

$u_{0j} \sim \mathcal{N}(0,\tau_{00}^2)$

$u_{1j} \sim \mathcal{N}(0,\tau_{11}^2)$

Average effect of LMX on sub perf across all groups

Moderating effect of team culture on the effect of LMX on sub perf

---

Limitations of HLM

Level 1 (e.g., subs) $y_{ij} = \beta_{0j} + \beta_{1j}\, x_{ij} + r_{ij}$

Level 2 (e.g., teams) $\beta_{0j} = \gamma_{00} + \gamma_{01}\, w_j + u_{0j}$
$\beta_{1j} = \gamma_{10} + \gamma_{11}\, w_j + u_{1j}$

Higher level variables explaining intercepts and slope of lower level regressions

Level 1 (e.g., subs) $y_k = \beta_0 + \beta_1\, x_{ik} + e_k$

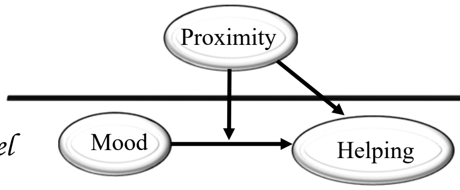
Team performance

Ability of team members

A Realistic Model

Team level

Individual level


Proximity

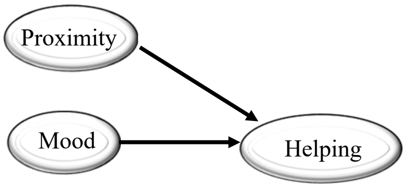
Mood

Helping

$$help_{ij} = \beta_{0j} + \beta_{1j}\ mood_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{00}\ prox_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11}\ prox_j + u_{1j}$$

Source: Hofmann, D.A. (1997). An Overview of the logic and rationale of hierarchical linear models, Journal of Management, 23(6), 723-744.

A single level approach


Proximity

Mood

Helping

$$help_{ij} = \beta_0 + \beta_1\ mood_{ij} + \beta_2\ prox_j + r_{ij}$$

Source: Hofmann, D.A. (1997). An Overview of the logic and rationale of hierarchical linear models, Journal of Management, 23(6), 723-744.

Mplus Regression program

```
TITLE:      Sample multi-level program on helping

DATA:       FILE = helping.txt;

VARIABLE:   NAMES = group help mood prox;

            USEVARIABLES = help mood prox;

MODEL:      help on mood prox;

OUTPUT:     TECH1;
```

$$\boxed{helping = \beta_0 + \beta_1\ mood + \beta_2\ proximity}$$

Mplus output

```
MODEL RESULTS
                                                    Two-Tailed
                   Estimate       S.E.    Est./S.E.    P-Value
 HELP    ON
   MOOD               3.976       0.087     45.915      0.000
   PROX               1.261       0.111     11.366      0.000

 Intercepts
   HELP               1.502       0.833      1.803      0.071

 Residual Variances
   HELP              40.962       1.832     22.359      0.000
```

$$helping = 1.502 + 3.976^{**}\ mood + 1.261^{**}\ proximity$$

The Null model (random intercept only)

$$help_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$

Baseline model or null model means that there is no predictor in the model at all.

What is β_{0j}? β_{0j} is the average helping behavior of each group.

What is r_{ij}? r_{ij} is the random error associated with each employee in each group.

What is γ_{00}? γ_{00} is the average helping of all employees in all groups (the grand mean).

What is u_{0j}? u_{0j} is the deviation of “average helping of group j employees” from the grand mean.

$$help_{ij} = \gamma_{00} + u_{0j} + r_{ij}$$

The helping behavior of ith employee in the jth group is the sum of :
- the average helping of all employees in all groups
- the deviation of the jth group mean from the grand mean
- the deviation of the ith individual from the jth group mean

Mplus Baseline Model

```
TITLE:      Sample multi-level program on helping
DATA:       FILE = helping.txt;
VARIABLE:   NAMES = group help mood prox;
            USEVARIABLES = help;
            WITHIN = ;
            BETWEEN = ;
            CLUSTER = group;
ANALYSIS:   TYPE = TWOLEVEL RANDOM;
MODEL:
        %WITHIN%
          help;

        %BETWEEN%
          help;

OUTPUT:     TECH1 TECH3;
```

$$help_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$

Mplus output

$$help_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$

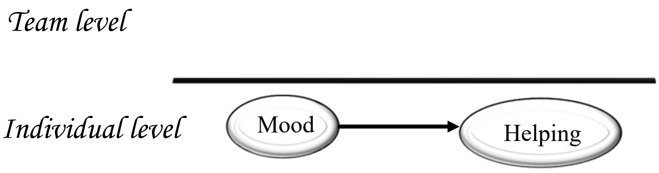
```
MODEL RESULTS
                                                   Two-Tailed
                     Estimate       S.E.   Est./S.E.   P-Value
Within Level
 Variances             σ^2
  HELP                31.757       1.763     18.011      0.000

Between Level
 Means                 γ_00
  HELP                31.394       1.410     22.269      0.000

 Variances             τ_00
  HELP                97.786      15.287      6.397      0.000
```

```
Intraclass
   Variable  Correlation
   HELP        0.755
```

$$\boxed{ICC = \frac{\tau_{00}}{\tau_{00} + \sigma^2}} = \frac{97.786}{97.786+31.757} = .755$$

What if ICC →0?

Fixed coefficient model



$$help_{ij} = \beta_{0j} + \beta_{1j}\ mood_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} \longleftarrow$$

The effect of mood on helping is the same across all groups

Mplus Fixed coefficient model

```
TITLE:       Sample multi-level program on helping
DATA:        FILE = helping.txt;
VARIABLE:    NAMES = group help mood prox;
             USEVARIABLES = help mood;
             WITHIN = mood;
             CLUSTER = group;
ANALYSIS:    TYPE = TWOLEVEL RANDOM;
MODEL:
        %WITHIN%
        help on mood;
        help;

        %BETWEEN%
        help;

OUTPUT:      TECH1 TECH3;
```

$$help_{ij} = \beta_{0j} + \beta_{1j}\ mood_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10}$$

helping affected by mood

allow mean difference across group

Estimate the mean of each group at Level 2

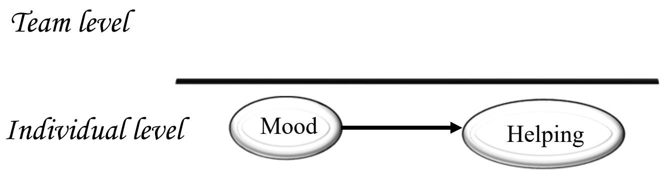
Mplus HLM output

$$help_{ij} = \beta_{0j} + \beta_{1j}\ mood_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10}$$

	Estimate	S.E.	Est./S.E.	
Within Level				
HELP ON	γ_{10}			
MOOD	2.999	0.068	44.230	0.000
Residual Variances	σ^2			
HELP	5.963	0.315	18.938	0.000
Between Level				
Means	γ_{00}			
HELP	31.394	1.410	22.269	0.000
Variances	τ_{00}			
HELP	99.076	15.275	6.486	0.000

The within variable is helping, σ^2 dropped from 31.757 to 5.963

Random coefficient model



$$help_{ij} = \beta_{0j} + \beta_{1j}\ mood_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j} \leftarrow$$

The effect of mood on helping is allowed to vary across groups

---

Mplus Random coefficient model

```
TITLE:      Sample multi-level program on helping
DATA:       FILE = helping.txt;
VARIABLE:   NAMES = group help mood prox;
            USEVARIABLES = help mood;
            WITHIN = mood;
            CLUSTER = group;
ANALYSIS:   TYPE = TWOLEVEL RANDOM;
MODEL:
        %WITHIN%
        help;
        s | help on mood;

        %BETWEEN%
        help;
        s;
        help WITH s;
OUTPUT:     TECH1 TECH3;
```

$$help_{ij} = \beta_{0j} + \beta_{1j}\ mood_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$

- we label the first-level effect of mood on help (i.e. β_{1j}) as s;
- mean help of each group (i.e. β_{0j}) as represented by "help" at the %BETWEEN% statement is estimated automatically.
- The intercept (β_{0j}) and slope (β_{1j}) in level-2 are correlated.

Mplus HLM output

$$help_{ij} = \beta_{0j} + \beta_{1j}\ mood_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$

```
                        Estimate        S.E.     Est./S.E.
 Within Level

  Residual Variances     σ2
    HELP                 5.611         0.304      18.452      0.000

 Between Level

  HELP     WITH
    S                   -0.105         0.443      -0.237      0.813

                         γ00
  Means
    HELP                13.837    γ10  0.959      14.436      0.000
    S                    3.013         0.069      43.935      0.000

                         τ00
  Variances
    HELP                41.681         7.899       5.277      0.000
    S                    0.125    τ11  0.037       3.360      0.001
```

The within variable is helping, σ^2 dropped from 5.963 to 5.611

Original s fixed effect estimate was 2.999

Variance of s (τ_{11}) is highly significant

HLM : Centering

No centering

The helping of the i^{th} sub in the j^{th} group

$$x_{ij}$$

$$\bar{x}_{\cdot\cdot}$$

Grand mean centering

Mean helping of all sub in the whole sample

$$\bar{x}_{\cdot j}$$

Group mean centering

Mean helping of all sub in the j^{th} group

Three common centering

Expected value of y_{ij} when x_{ij} is 0

Expected change in y_{ij} given a unit change in x_{ij}

$$\boxed{Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij}}$$

Expected value of y_{ij} when x_{ij} is equal to mean x_{ij} of the jth group

Expected change in y_{ij} given a unit deviation of x_{ij} from mean x_{ij} of the same group

Job performance of the ith sub in the jth team

$$\boxed{Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}}$$

Expected value of y_{ij} when x_{ij} is equal to the overall mean x_{ij}

Expected change in y_{ij} given a unit deviation of x_{ij} from the overall mean x_{ij}

$$\boxed{Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{..}\right) + r_{ij}}$$

Group Mean Centering

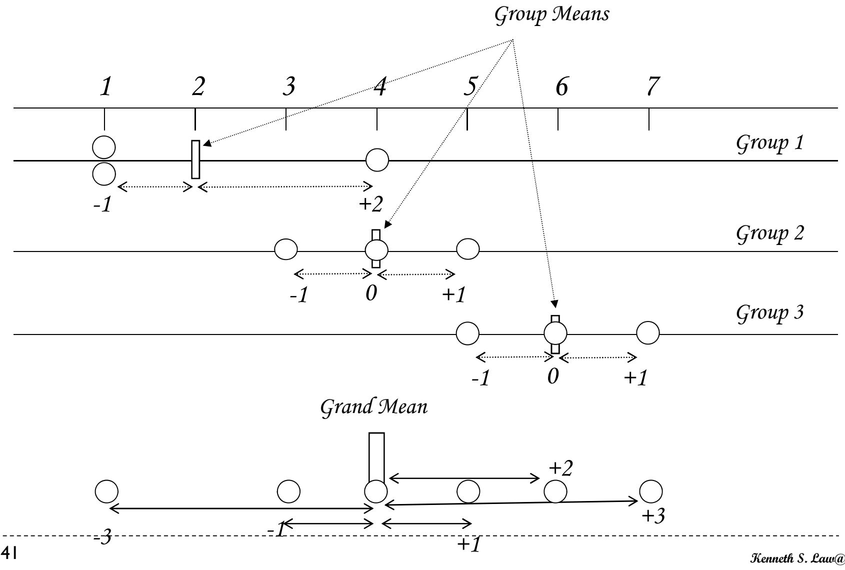
$$\boxed{Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}}$$

Group	Person	*x*	*x'*	*y*
1	1	2	-1	3
1	2	3	0	3
1	3	3	0	4
1	4	4	1	4
2	1	3	-1	2
2	2	4	0	3
2	3	5	1	4
3	1	2	-.5	4
3	2	3	.5	3

$\bar{x}_1=3$

$\bar{x}_2=4$

Grand-mean vs. Group-mean centering

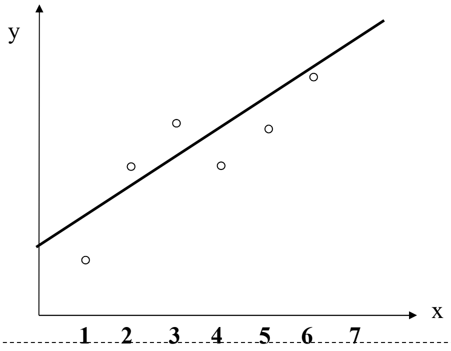


No Centering

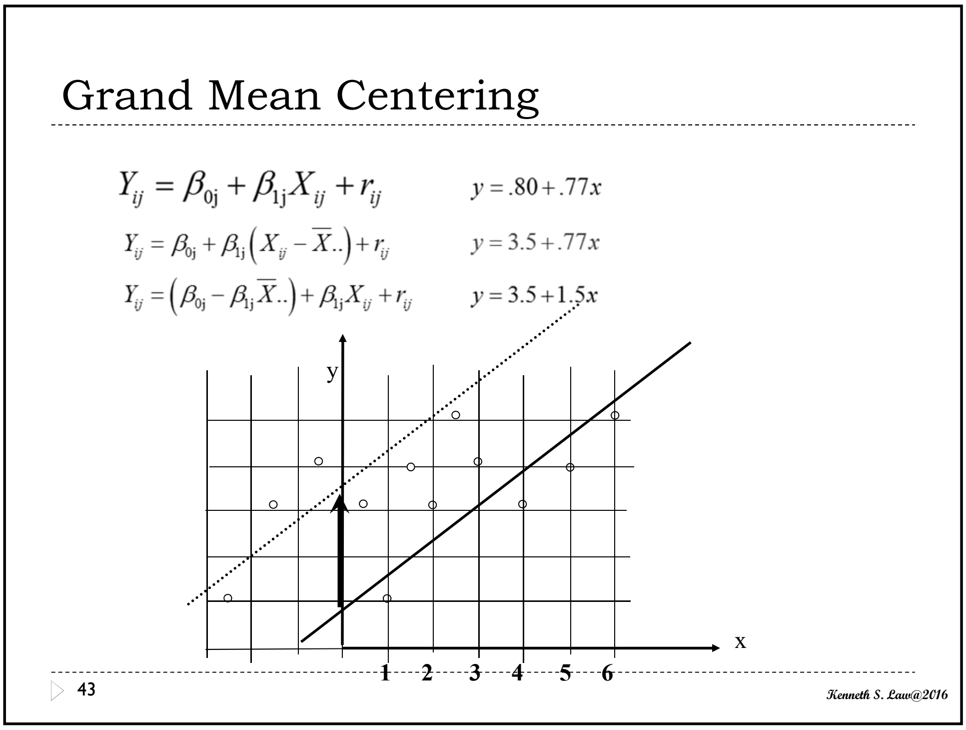
$Y_{ij} = \beta_{0j} + \beta_{1j} X_{ij} + r_{ij}$ $\quad y = .80 + .77x \quad y = .878x$

$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}..\right) + r_{ij}$ $\quad y = 3.5 + .77x \quad y = .878x$

$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}._{j}\right) + r_{ij}$ $\quad y = 3.5 + 1.5x \quad y = .816x$



y	Raw x	Grand	Group
1	1	-2.5	-1
3	2	-1.5	0
4	3	-0.5	1
3	4	0.5	-1
4	5	1.5	0
6	6	2.5	1
b0	0.80	3.50	3.50
b1	0.77	0.77	1.50

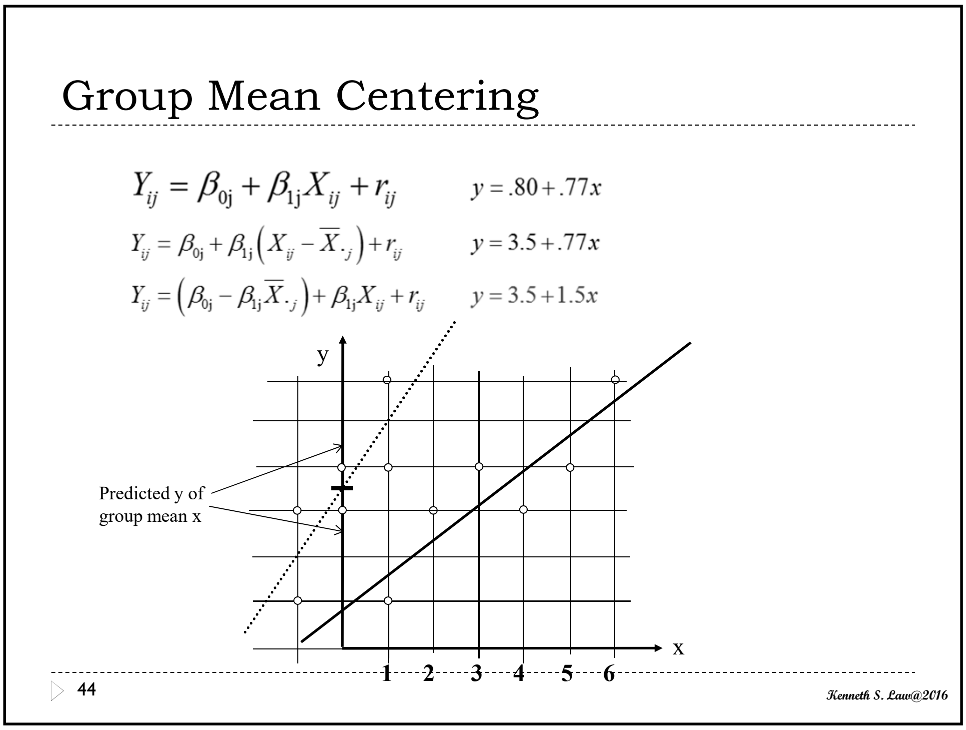


Grand Mean Centering

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij} \qquad y = .80 + .77x$$

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}..\right) + r_{ij} \qquad y = 3.5 + .77x$$

$$Y_{ij} = \left(\beta_{0j} - \beta_{1j}\overline{X}..\right) + \beta_{1j}X_{ij} + r_{ij} \qquad y = 3.5 + 1.5x$$



Group Mean Centering

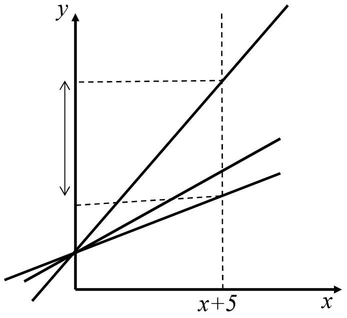
$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij} \qquad y = .80 + .77x$$

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}._{j}\right) + r_{ij} \qquad y = 3.5 + .77x$$

$$Y_{ij} = \left(\beta_{0j} - \beta_{1j}\overline{X}._{j}\right) + \beta_{1j}X_{ij} + r_{ij} \qquad y = 3.5 + 1.5x$$

Effect of shifting on intercept & variance

b_{0j} would be different if $x' = x+5$



For the original x values, the three groups would have exactly the same intercept.

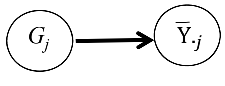
- *Not only the mean values of b_{0j} would be different, the variance of b_{0j}, i.e., τ_{00} would also be different with different centering options*
- *The same is true for b_{1j} and τ_{11} & τ_{01}.*

Group Mean Centering

- when group mean centering is adopted, the level-1 intercept variance is equal to the between group variance in the outcome measure.
- As a result, the level-2 regression coefficients, under group mean centering, simply represent the group level relationship between the level-2 predictor and the outcome variable of interest (i.e., the relationship between the level-2 predictor, X_j, and Y_j).

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{\cdot j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} G_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11} G_j + u_{0j}$$

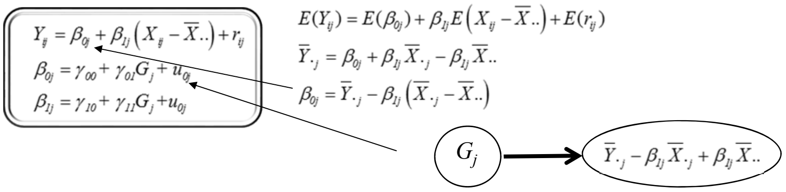
$$E(Y_{ij}) = E(\beta_{0j}) + \beta_{1j} E\left(X_{ij} - \overline{X}_{\cdot j}\right) + E(r_{ij})$$
$$\overline{Y}_{\cdot j} = \beta_{0j}$$
$$\beta_{0j} = \overline{Y}_{\cdot j}$$



Grand Mean Centering

- when grand mean centering is adopted, the variance in the intercept term represents the between group variance in the outcome measure adjusted for the level-1 predictor(s).

- The level-2 regression coefficients represent the group level relationship between the level-2 predictor and the outcome variable less the influence of the level-1 predictor(s).



$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{..}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01}G_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11}G_j + u_{0j}$$

$$E(Y_{ij}) = E(\beta_{0j}) + \beta_{1j}E\left(X_{ij} - \overline{X}_{..}\right) + E(r_{ij})$$
$$\overline{Y}_{\cdot j} = \beta_{0j} + \beta_{1j}\overline{X}_{\cdot j} - \beta_{1j}\overline{X}_{..}$$
$$\beta_{0j} = \overline{Y}_{\cdot j} - \beta_{1j}\left(\overline{X}_{\cdot j} - \overline{X}_{..}\right)$$

$$G_j \longrightarrow \overline{Y}_{\cdot j} - \beta_{1j}\overline{X}_{\cdot j} + \beta_{1j}\overline{X}_{..}$$

Mplus Random coefficient model

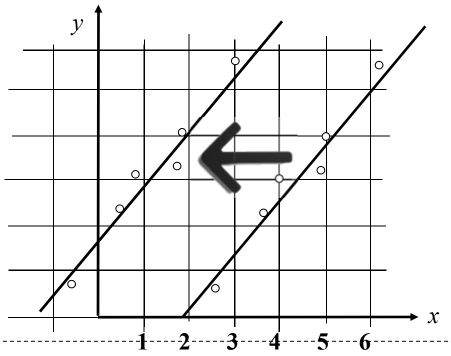
```
TITLE:      Sample multi-level program on helping
DATA:       FILE = helping.txt;
DEFINE:     center help (grandmean);
VARIABLE:   NAMES = group help mood prox;
            USEVARIABLES = help mood;
            WITHIN = mood;
            CLUSTER = group;
ANALYSIS:   TYPE = TWOLEVEL RANDOM;
MODEL:
        %WITHIN%
        help;
        s | help on mood;

        %BETWEEN%
        help;
        s;
        help WITH s;
OUTPUT:     TECH1 TECH3;
```

$$help_{ij} = \beta_{0j} + \beta_{1j}\left(mood_{ij} - \overline{mood}_{..}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$

Grand Mean Centering

$$Y_{ij} = \beta_{0j} + \beta_{1j} X_{ij} + r_{ij}$$

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}..\right) + r_{ij}$$



Mplus HLM output

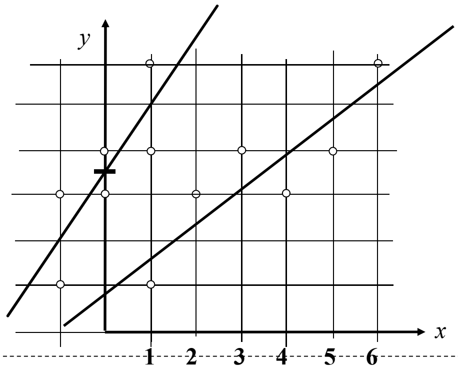
$$help_{ij} = \beta_{0j} + \beta_{1j}\left(mood_{ij} - \overline{mood}..\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$

	No Centering		Grand Mean centering	
	Estimate	P-Value	Estimate	P-Value
Within Level				
Residual Variances				
HELP	5.611	0.000	5.611	0.000
Between Level				
HELP WITH				
S	-0.105	0.813	0.622	0.156
Means				
HELP	13.837	0.000	31.428	0.000
S	3.013	0.000	3.013	0.000
Variances				
HELP	41.681	0.000	44.702	0.000
S	0.125	0.001	0.125	0.001

Group Mean Centering

$$Y_{ij} = \beta_{0\mathrm{j}} + \beta_{1\mathrm{j}} X_{ij} + r_{ij} \qquad y = .80 + .77x$$

$$Y_{ij} = \beta_{0\mathrm{j}} + \beta_{1\mathrm{j}}\left(X_{ij} - \overline{X}_{\cdot j}\right) + r_{ij} \qquad y = 3.5 + 1.5x$$



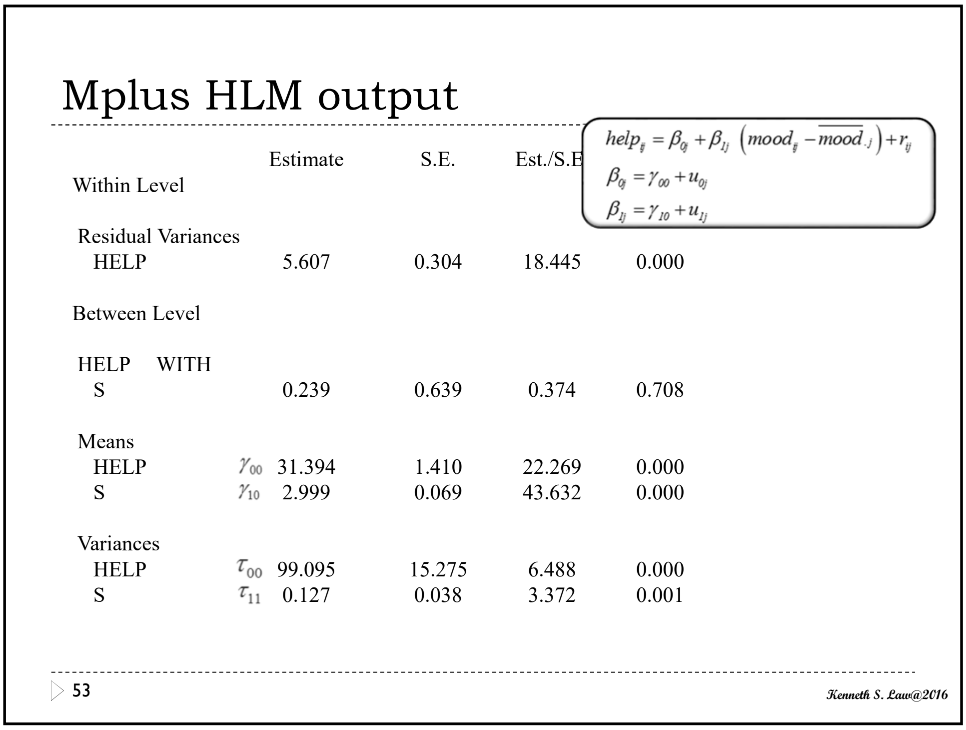
Mplus Random coefficient model

```
TITLE:      Sample multi-level program on helping
DATA:       FILE = helping.txt;
DEFINE:     center help (groupmean);
VARIABLE:   NAMES = group help mood prox;
            USEVARIABLES = help mood;
            WITHIN = mood;
            CLUSTER = group;
ANALYSIS:   TYPE = TWOLEVEL RANDOM;
MODEL:
        %WITHIN%
        help;
        s | help on mood;

        %BETWEEN%
        help;
        s;
        help WITH s;
OUTPUT:     TECH1 TECH3;
```

$$help_{ij} = \beta_{0j} + \beta_{1j}\left(mood_{ij} - \overline{mood}_{\cdot j}\right) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + u_{1j}$$



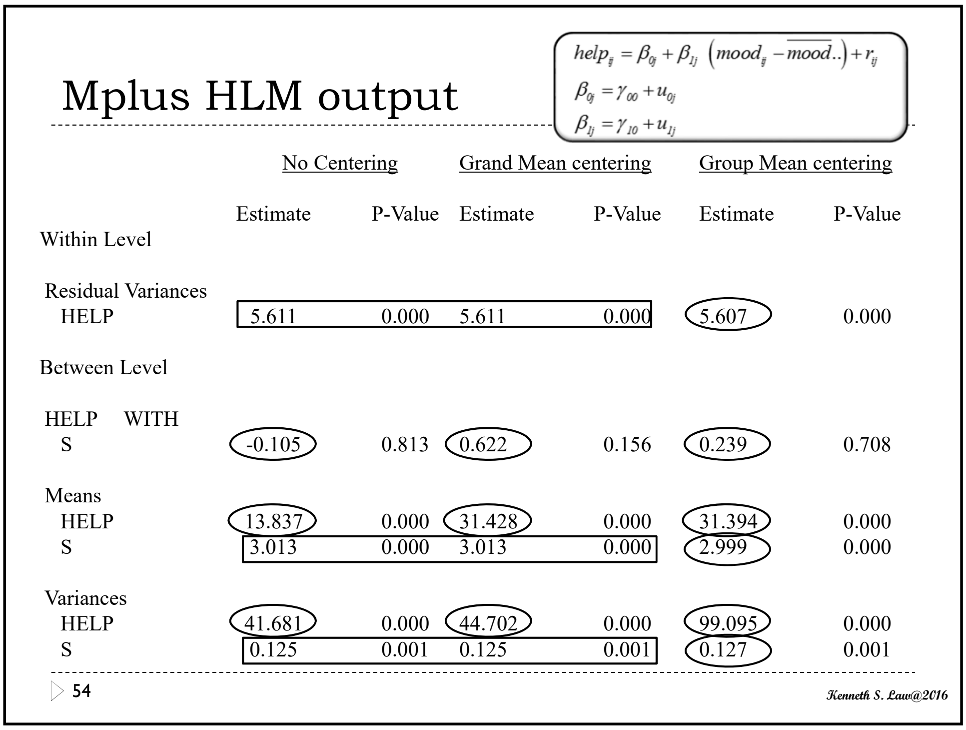
Mplus HLM output

$$help_{ij} = \beta_{0j} + \beta_{1j}\left(mood_{ij} - \overline{mood}_{.j}\right) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + u_{1j}$$

		Estimate	S.E.	Est./S.E	
Within Level					
Residual Variances					
HELP		5.607	0.304	18.445	0.000
Between Level					
HELP WITH					
S		0.239	0.639	0.374	0.708
Means					
HELP	γ_{00}	31.394	1.410	22.269	0.000
S	γ_{10}	2.999	0.069	43.632	0.000
Variances					
HELP	τ_{00}	99.095	15.275	6.488	0.000
S	τ_{11}	0.127	0.038	3.372	0.001



Mplus HLM output

$$help_{ij} = \beta_{0j} + \beta_{1j}\left(mood_{ij} - \overline{mood}_{..}\right) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + u_{1j}$$

	No Centering		Grand Mean centering		Group Mean centering	
	Estimate	P-Value	Estimate	P-Value	Estimate	P-Value
Within Level						
Residual Variances						
HELP	5.611	0.000	5.611	0.000	5.607	0.000
Between Level						
HELP WITH						
S	-0.105	0.813	0.622	0.156	0.239	0.708
Means						
HELP	13.837	0.000	31.428	0.000	31.394	0.000
S	3.013	0.000	3.013	0.000	2.999	0.000
Variances						
HELP	41.681	0.000	44.702	0.000	99.095	0.000
S	0.125	0.001	0.125	0.001	0.127	0.001

A recap

Examples of different HLM models

Model 1

Random errors

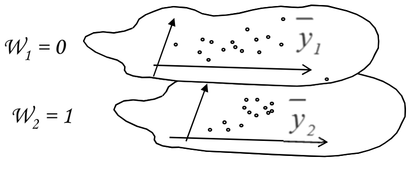
$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$$

Grand mean

Random group effects



Y_{ij} is affected by:

(1) A grand mean;
(2) The random (group) effects
(3) Random (individual) errors

One-way ANOVA random effects

random errors

Level 1 $$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$

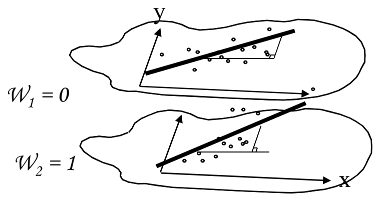
Level 2 $$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$

random effects

$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$$

$$\boxed{\begin{aligned} Y_{ij} &= \beta_{0j} + r_{ij} \\ \beta_{0j} &= \gamma_{00} + u_{0j} \end{aligned}}$$

Model 2



Random errors

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$

Grand mean

Random effects

$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$$

Y_{ij} is affected by:
(1) A grand mean;
(2) Random (group) effects
(3) A third variable (X) effects
(4) Random (individual) errors

Note: the X-effect (covariance effect) is constant. It is neither affected by another variable W, nor having a random component.

One-way ANCOVA random effects

Level 1 $y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + r_{ij}$

Level 2 $\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$

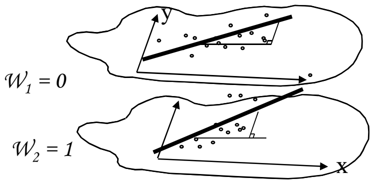
$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$

$$\boxed{\begin{aligned} y_{ij} &= \beta_{0j} + \beta_{1j} x_{ij} + r_{ij} \\ \beta_{0j} &= \gamma_{00} + u_{0j} \\ \beta_{1j} &= \gamma_{10} \end{aligned}}$$

random effects

No random effects

Model 3



Random errors

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$$

Simple regression models

Level 1 $Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$

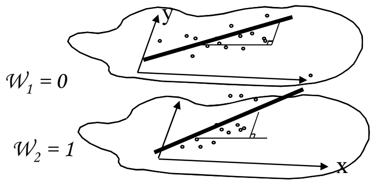
Level 2 $\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$

$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00}$$
$$\beta_{1j} = \gamma_{10}$$

Level-2 effect is not modeled, it reduces to Level-1 effect only.

Model 4



Random errors

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$

Grand mean

Random effects

Y_{ij} is affected by:
(1) A grand mean;
(2) Random (group) effects
(3) A third variable (X) effects
(4) Random (individual) errors

Mean-as-outcome regression

Level 1 $Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$

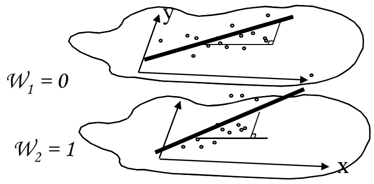
Level 2 $\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$

$$Y_{ij} = \beta_{0j} + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$

The ***means*** from each of many groups ***as an outcome*** to be predicted by group characteristics.

Model 5



Random errors

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$

Random effects

$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$$

Grand mean

Y_{ij} is affected by:
1. A grand mean;
2. Random (group) effects
3. A third variable (X) effects
4. Random (individual) errors

Intercepts-and-Slopes-as-outcome regression

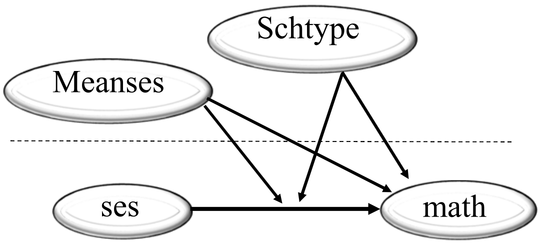
Level 1 $Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$

Level 2 $\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$

$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$

- Both the ***intercepts*** and ***slopes*** as ***outcomes*** to be ***regressed*** (predicted by W_j).
- Have you studied all W_j of interests, or do you want to generalize to other W_j s.

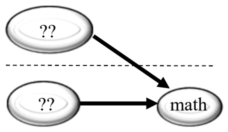
Another example



Model 1: Null model

```
TITLE:        Sample multi-level program on helping (hlm_sem1)

DATA:         FILE = hlm_sem.dat;

VARIABLE:     NAMES = schid minority female ses math size schtype meanses;
              USEVARIABLES = math;
              WITHIN = ;        ! level 1 variables here (none)
              BETWEEN = ;       ! level 2 variables here (none)
              CLUSTER = schid;

ANALYSIS:   TYPE = TWOLEVEL RANDOM;

MODEL:
       %WITHIN%
              math;             ! no fixed effects
       %BETWEEN%
              math;             ! no predictors of intercept

OUTPUT:       TECH1 TECH3;
```

$$math_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$



Mplus output

$$math_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$

```
                         Estimate     S.E.    Est./S.E.    P-Value
 Within Level

  Variances
    MATH                  39.148      0.835     46.876       0.000

 Between Level

  Means
    MATH                  12.637      0.244     51.823       0.000

  Variances
    MATH                   8.562      1.057      8.100       0.000
```

error variance level-1 σ^2 → 39.148

γ_{00} → 12.637

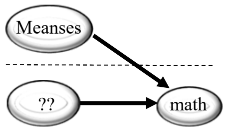
τ_{00} → 8.562, error variance level-2

The within variable is MATH, σ^2 = 39.148. This is the variance of r_{ij}, which is constant across level-2 units.

Model 2: A level-2 predictor

```
TITLE:          Sample multi-level program on helping (hlm_sem2)

DATA:           FILE = hlm_sem.dat;

VARIABLE:       NAMES = schid minority female ses math size schtype meanses;
                USEVARIABLES = math meanses;
                WITHIN = ;
                BETWEEN = meanses;
                CLUSTER = schid;

ANALYSIS:   TYPE = TWOLEVEL RANDOM;

MODEL:
        %WITHIN%
                math;
        %BETWEEN%
                math ON meanses; ! Level 2 predictor of mean ses

OUTPUT:         TECH1 TECH3;
```

$$math_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} Meanses + u_{0j}$$



Mplus output

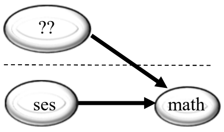
$$math_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} Meanses + u_{0j}$$

```
                      Estimate         S.E.     Est./S.E.    P-Value
Within Level

 Variances                 σ²
  MATH                39.157          0.836      46.838       0.000

Between Level
                      Meanses can be used to predict mean Math between groups
 MATH      ON
  MEANSES              5.863          0.321      18.276       0.000

 Intercepts
  MATH                12.650          0.148      85.575       0.000

 Residual Variances           τ00 dropped from 8.562 to 2.598
  MATH                 2.598          0.467       5.558       0.000
```

Model 3: A level-1 predictor

```
TITLE:        Sample multi-level program on helping (hlm_sem3)

DATA:         FILE = hlm_sem.dat;

VARIABLE:     NAMES = schid minority female ses math size schtype meanses;
              USEVARIABLES = math ses;
              WITHIN = ses;   ! Level 1 predictor of math
              BETWEEN = ;
              CLUSTER = schid;

ANALYSIS:   TYPE = TWOLEVEL RANDOM;

MODEL:
        %WITHIN%
              math; ! Mean (intercept) of math for each group
              s | math ON ses;  ! slope for each group
        %BETWEEN%
              math;      ! nothing predicts intercept
              s;            ! nothing predicts slope
              math with s;   ! Covariance between intercept and slope

OUTPUT:       TECH1 TECH3;
```

$$math_{ij} = \beta_{0j} + \beta_{1j} ses + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11} + u_{1j}$$



Mplus output

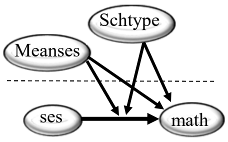
$$math_{ij} = \beta_{0j} + \beta_{1j} ses + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11} + u_{1j}$$

σ^2 dropped from 39.148 to 36.820

```
                      Estimate       S.E.     Est./S.E.    P-Value
Within Level

 Residual Variances
    MATH               36.831        0.725     50.779       0.000

Between Level
MATH    WITH
    S                  -0.167        0.326     -0.513       0.608

 Means
    MATH               12.666        0.203     62.281       0.000
    S                   2.394        0.128     18.646       0.000

 Variances
    MATH                4.781        0.756      6.324       0.000
    S                   0.402        0.240      1.670       0.095
```

Model 4: Both level-1 & level-2 predictor

```
TITLE:       Sample multi-level program on helping (hlm_sem4)

DATA:        FILE = hlm_sem.dat;
DEFINE:      center ses (groupmean);
VARIABLE:    NAMES = schid minority female ses math size schtype meanses;
             USEVARIABLES = math ses schtype meanses;
             WITHIN = ses;
             BETWEEN = schtype meanses;
             CLUSTER = schid;

ANALYSIS:  TYPE = TWOLEVEL RANDOM;

MODEL:
      %WITHIN%
             math;
             s | math ON ses;
      %BETWEEN%
             math ON schtype meanses;
             s ON schtype meanses;
             math WITH s
```



$$math_{ij} = \beta_{0j} + \beta_{1j}\left(ses_{ij} - \overline{ses}_{.j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01}schtype_j + \gamma_{02}meanses_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11}schtype_j + \gamma_{12}meanses_j + u_{1j}$$

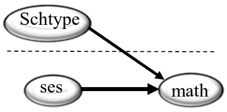
```
OUTPUT:      TECH1 TECH3;
```

Mplus output

```
                        Estimate          S.E.      Est./S.E.    P-Value
Within Level           σ2 dropped from
 Residual Variances     39.148 to 36.720
   MATH             σ2  36.720          0.721

Between Level
 S        ON
   SCHTYPE          γ11  -1.640         0.238        -6.905       0.000
   MEANSES          γ12   1.033         0.333         3.100       0.002

 MATH       ON
   SCHTYPE          γ01   1.227         0.308         3.982       0.000
   MEANSES          γ02   5.332         0.336        15.871       0.000

 MATH     WITH
   S                      0.200         0.192         1.041       0.298

 Intercepts
   MATH             γ00  12.096         0.174        69.669       0.000
   S                γ10   2.938         0.147        19.986       0.000

 Residual Variances
   MATH             τ00   2.316         0.414         5.591       0.000
   S                τ11   0.071         0.201         0.352       0.725
```

$$math_{ij} = \beta_{0j} + \beta_{1j}\left(ses_{ij} - \overline{ses}_{.j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01}schtype_j + \gamma_{02}meanses_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11}schtype_j + \gamma_{12}meanses_j + u_{1j}$$

Model 5: Fixed effect for slope



```
TITLE:       Sample multi-level program on helping (hlm_sem4)

DATA:        FILE = hlm_sem.dat;
DEFINE:      center ses (groupmean);
VARIABLE:    NAMES = schid minority female ses math size schtype meanses;
             USEVARIABLES = math ses schtype;
             WITHIN = ses;
             BETWEEN = schtype;
             CLUSTER = schid;

ANALYSIS:  TYPE = TWOLEVEL RANDOM;

MODEL:
       %WITHIN%
             math ON ses;    Note:  NO  s| math on ses for fixed effect
       %BETWEEN%
             math ON schtype;
             ! Nothing to predict s since it is a fixed effect
             ! NO math WITH s since s will not change across level

OUTPUT:      TECH1 TECH3;
```

$$math_{ij} = \beta_{0j} + \beta_{1j}\left(ses_{ij} - \overline{ses}_{.j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01}schtype_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10}$$

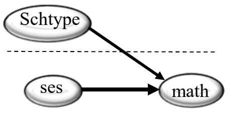
Mplus output

$$math_{ij} = \beta_{0j} + \beta_{1j}\left(ses_{ij} - \overline{ses}_{.j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01}schtype_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10}$$

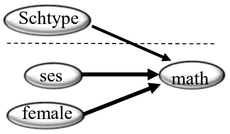
```
                          Estimate        S.E.       Est./S.E.     P-Value
 Within Level
  MATH       ON
    SES             γ10   2.191           0.129      16.938        0.000

  Residual Variances
    MATH            σ2    37.008          0.715      51.771        0.000

 Between Level
  MATH       ON
    SCHTYPE         γ01   2.805           0.436      6.434         0.000

  Intercepts
    MATH            γ00   11.393          0.292      38.959        0.000

  Residual Variances
    MATH            τ00   6.643           0.869      7.645         0.000
```

Model 6: Random and fixed effect for slope

```
TITLE:        Sample multi-level program on helping (hlm_sem4)

DATA:         FILE = hlm_sem.dat;
DEFINE:       center ses (groupmean);
VARIABLE:     NAMES = schid minority female ses math size schtype meanses;
              USEVARIABLES = math ses female schtype;
              WITHIN = ses female;
              BETWEEN = schtype;
              CLUSTER = schid;

ANALYSIS:   TYPE = TWOLEVEL RANDOM;

MODEL:
        %WITHIN%
              s | math ON ses;    Note:  NO  s| math on ses for fixed effect
              math ON female;
        %BETWEEN%
              math ON schtype;
              s;          ! Nothing to predict s
              math WITH s;

OUTPUT:       TECH1 TECH3;
```



$$math_{ij} = \beta_{0j} + \beta_{1j} ses + \beta_{2j} female + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} schtype_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$
$$\beta_{2j} = \gamma_{20}$$



Mplus output

$$math_{ij} = \beta_{0j} + \beta_{1j} ses + \beta_{2j} female + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} schtype_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$
$$\beta_{2j} = \gamma_{20}$$

```
                             Estimate       S.E.
 Within Level
  MATH       ON             Fixed effect
   FEMALE                  -1.199         0.182       -6.599       0.000

 Residual Variances
   MATH              σ2    36.602         0.714       51.262       0.000

 Between Level
  MATH       ON
   SCHTYPE           γ01    2.548         0.405        6.286       0.000

  MATH     WITH
   S                        0.704         0.347        2.030       0.042

  Means
   S                 γ10    2.352         0.125       18.775       0.000

  Intercepts
   MATH              γ00   12.092         0.302       39.997       0.000

  Variances
   S                 τ11    0.360         0.228        1.575       0.115

  Residual Variances
   MATH              τ00    3.626         0.629        5.766       0.000
```

Mplus programming

Mplus Programming: random intercept

```
TITLE:    two-level regression with a
random intercept and an observed covariate
DATA:     FILE = ex9.1a.dat;
VARIABLE:
NAMES = y x w xm clus;
WITHIN = x;          Level 1
BETWEEN = w xm;          Level 2
CLUSTER = clus;
DEFINE:
    CENTER x (GRANDMEAN);
ANALYSIS:
    TYPE = TWOLEVEL;
MODEL:
  %WITHIN%
     y ON x;
  %BETWEEN%
     y ON w xm;
```

(1) No need to declare whether a y variable is between or within.
(2) If you do not state whether a variable is WITHIN or BETWEEN, Mplus assumes that it has <u>both</u> between and within variances.

Grouping variable 1, 2, 3, 4, …

Multilevel model with random intercept

$$Y_{ij} = \beta_{0j} + \beta_1 X_{ij} + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + \gamma_{02} \bar{x}_{.j} + u_{0j}$$

Mplus Output

```
SUMMARY OF DATA

     Number of clusters                   110

    Average cluster size        9.091

     Estimated Intraclass Correlations for the Y
Variables

                Intraclass
    Variable  Correlation

    Y           0.570
```

Mplus Output

```
SUMMARY OF DATA

     Number of clusters                   110

    Average cluster size        9.091

     Estimated Intraclass Correlations for the Y
Variables

                Intraclass
    Variable  Correlation

    Y           0.570

THE MODEL ESTIMATION TERMINATED
NORMALLY
```

Mplus Output

```
MODEL FIT INFORMATION

Number of Free Parameters                    6

Loglikelihood

        H0 Value                     -1525.938
        H0 Scaling Correction Factor    0.9402
          for MLR
        H1 Value                     -1525.938
        H1 Scaling Correction Factor    0.9402
          for MLR

Information Criteria

        Akaike (AIC)                  3063.876
        Bayesian (BIC)                3093.322
        Sample-Size Adjusted BIC      3074.266
          (n* = (n + 2) / 24)

Chi-Square Test of Model Fit

        Value                           0.000*
        Degrees of Freedom                   0
        P-Value                         0.0000
        Scaling Correction Factor       1.0000
          for MLR
```

```
RMSEA (Root Mean Square Error Of Approximation)

        Estimate                        0.000

CFI/TLI

        CFI                             1.000
        TLI                             1.000

Chi-Square Test of Model Fit for the Baseline Model

        Value                         491.881
        Degrees of Freedom                  3
        P-Value                        0.0000

SRMR (Standardized Root Mean Square Residual)

        Value for Within                0.000
        Value for Between               0.000
```

Mplus Output

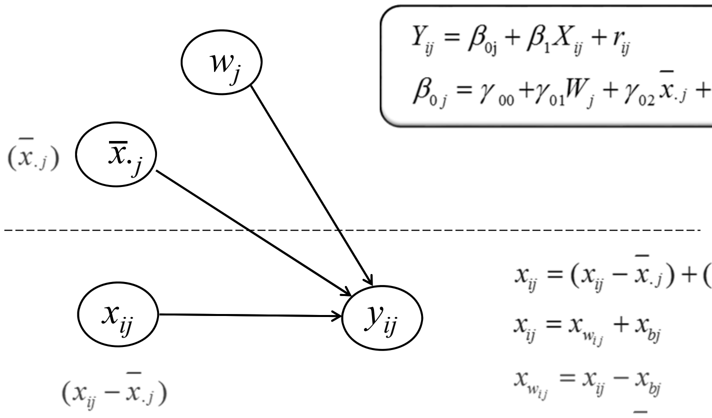
```
MODEL RESULTS

                                          Two-Tailed
                 Estimate    S.E.  Est./S.E.    P-Value

Within Level

 Y        ON       β1
    X              0.724    0.033     22.118      0.000

 Residual Variances  σ2
    Y              1.022    0.041     25.117      0.000

Between Level

 Y        ON       γ01
    W              0.570    0.108      5.305      0.000
    XM             0.976    0.160      6.107      0.000
                   γ02
 Intercepts
    Y              1.991    0.080     24.804      0.000
                  γ00
 Residual Variances
    Y              0.571    0.088      6.486      0.000
                  τ00
```

$$Y_{ij} = \beta_{0j} + \beta_1 X_{ij} + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + \gamma_{02} \bar{x}_{.j} + u_{0j}$$

Partitioning within and between variance

$$Y_{ij} = \beta_{0j} + \beta_1 X_{ij} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + \gamma_{02} \bar{x}_{.j} + u_{0j}$$



$$x_{ij} = (x_{ij} - \bar{x}_{.j}) + (\bar{x}_{.j})$$
$$x_{ij} = x_{w_{ij}} + x_{bj}$$
$$x_{w_{ij}} = x_{ij} - x_{bj}$$
$$x_{w_{ij}} = x_{ij} - \bar{x}_{.j}$$

Grou mean centering

Hypothetical data

innovation performance

Group	x	Mean x	y
1	3	4	5
1	4	4	5
1	5	4	5
2	1	2	3
2	2	2	3
2	3	2	3
3	2	3	4
3	3	3	4
3	4	3	4

$$r_{\bar{x}y} = 1.0 \quad r_{xy} = .71$$

Partitioning within and between variance

Note: the data does not have xm (we estimate xm from the data)

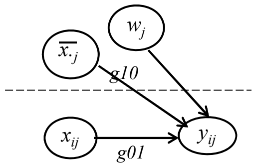
```
TITLE: Partitioning variances
DATA: FILE = ex9.1b.dat;
VARIABLE: NAMES = y x w clus;
BETWEEN = w;
CLUSTER = clus;
DEFINE: CENTER = x (GRANDMEAN);
ANALYSIS: TYPE = TWOLEVEL;
MODEL:
%WITHIN%
   y ON x (g01);
%BETWEEN%
   y ON w
   x (g10);
MODEL CONSTRAINT:
NEW(diff);
diff = g10 – g01;
```

We do not declare x as WITHIN

Same random intercept model

Effect of within group x on y

Effect of mean x on y



Mplus Programming: random intercept

```
TITLE:    two-level regression with a
random intercept and an observed covariate
DATA:     FILE = ex9.1a.dat;
VARIABLE:
NAMES = y x w xm clus;
WITHIN = x;
BETWEEN = w xm;
CLUSTER = clus
DEFINE:
   CENTER x (GRANDMEAN);
ANALYSIS:
   TYPE = TWOLEVEL;
MODEL:
  %WITHIN%
    y ON x;
  %BETWEEN%
    y ON w xm;
```

(1) No need to declare whether a y variable is between or within.

(2) If you do not state whether a variable is WITHIN or BETWEEN, Mplus assumes that it has <u>both</u> between and within variances.

Full HLM model

Random slope and intercept

Slope and intercept as outcome regression

Mplus : random slope & intercept

```
TITLE:    two-level regression with a
random intercept and an observed covariate
DATA:     FILE = ex9.2a.dat;
VARIABLE:
NAMES = y x w xm clus;
WITHIN = x;
BETWEEN = w xm;
CLUSTER = clus;
DEFINE:  CENTER x (GRANDMEAN);
ANALYSIS: TYPE = TWOLEVEL RANDOM;
MODEL:
  %WITHIN%
     s | y ON x;
  %BETWEEN%
     y s ON w xm;
     y WITH s
```

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + \gamma_{02}\bar{x}_{.j} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + \gamma_{12}\bar{x}_{.j} + u_{1j}$$

S is the slope of x→y for each group

w and xm affect both the intercept y and the slope s

The random slope is correlated with the random intercept

Mplus Output

```
MODEL RESULTS
                                    Two-Tailed
              Estimate     S.E.  Est./S.E.    P-Value

Within Level
Residual Variances  σ2
   Y              1.032     0.047    22.126     0.000

Between Level
S         ON
  W       γ11   0.396     0.097     4.063     0.000
  XM      γ12   0.542     0.136     4.001     0.000

Y         ON
  W       γ01   0.874     0.118     7.387     0.000
  XM      γ02   1.345     0.164     8.186     0.000

Y        WITH
  S             0.306     0.067     4.572     0.000

Intercepts
  Y       γ00   2.113     0.088    24.021     0.000
  S       γ10   1.039     0.075    13.813     0.000

Residual Variances
  Y       τ00   0.606     0.101     5.982     0.000
  S       τ11   0.334     0.056     5.932     0.000
```

No slope or intercept estimate at the within level because they are random

$$Y_{ij} = \beta_{0j} + \beta_{1j} X_{ij} + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + \gamma_{02} \bar{x}_{\cdot j} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + \gamma_{12} \bar{x}_{\cdot j} + u_{1j}$$

Correlation of random slope and random intercept

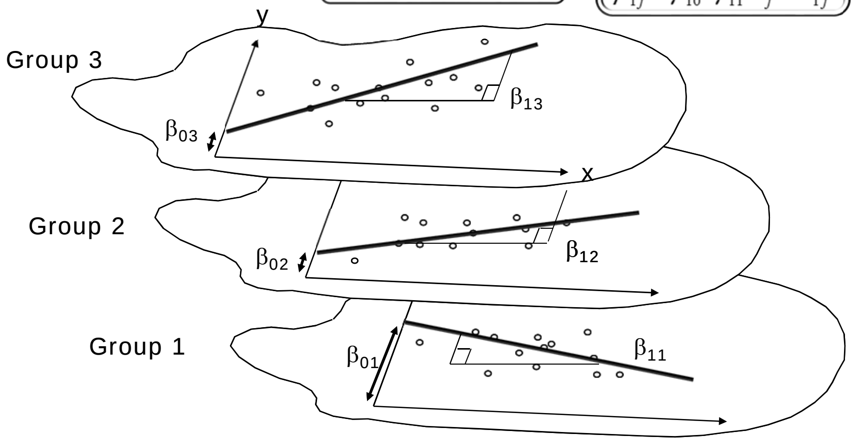
Estimation

Some issues on HLM estimates

Estimation in HLM

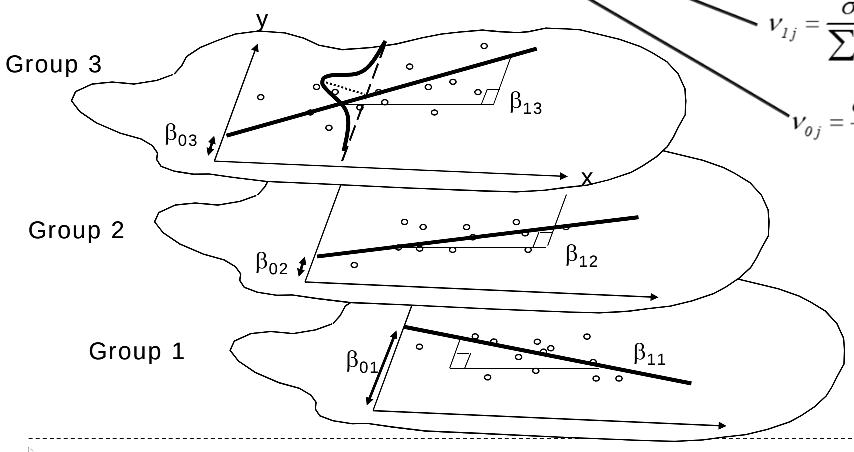
$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$$



Estimation in HLM

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij}$$

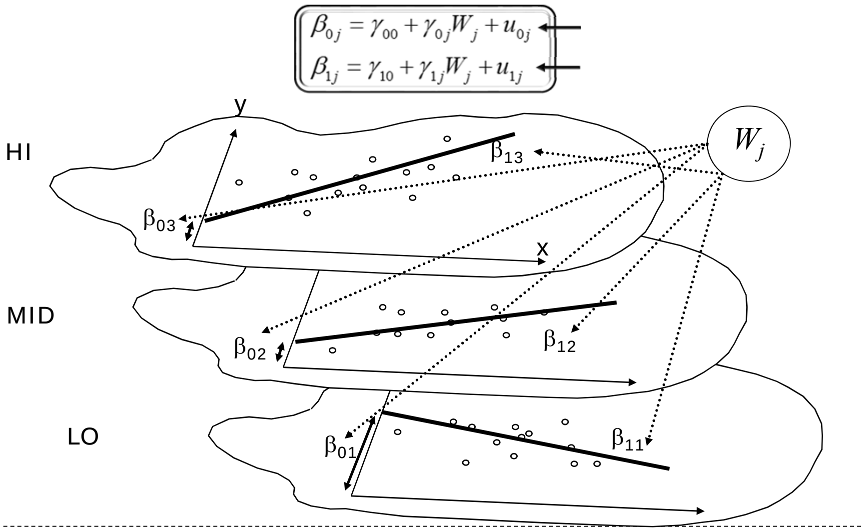


$$\sigma^2$$

$$v_{1j} = \frac{\sigma^2}{\sum x_{ij}^2}$$

$$v_{0j} = \frac{\sigma^2}{n_j}$$

Estimation in HLM

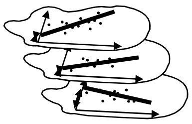


$$\beta_{0j} = \gamma_{00} + \gamma_{0j} W_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{1j} W_j + u_{1j}$$

Two sets of estimates (I)

- For a particular team (e.g., Team 2), one can simply run an OLS regression and estimate the Level 1 effect (β_{02} and β_{12}) using the Level 1 equation:

$$Y_{i2} = \beta_{02} + \beta_{12} X_{i2} + r_{i2}$$

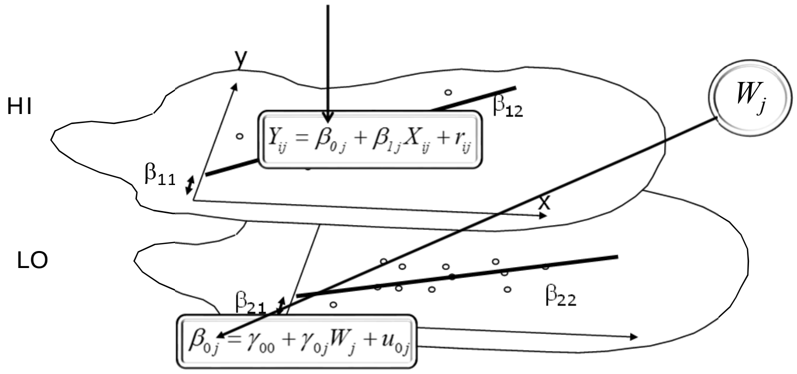


- After knowing β_{0k} and β_{1k} , one can then estimate γ_{00} , γ_{01} , γ_{10} , and γ_{11} using W_j in the following Level 2 equations:

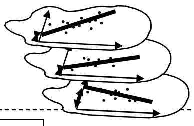
$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j} \quad (when\ j = 2)$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j} \quad (when\ j = 2)$$

- One then has two sets of β estimates, those estimated by Level 1 data and those estimated by Level 2 data.

Estimation in HLM



Two sets of estimates (II)



- βs as estimated using Level 1 data $\boxed{Y_{ijk} = \beta_{0k} + \beta_{1k} X_{ijk} + r_{ijk}}$
- βs as estimated using Level 2 data
$$\boxed{\begin{aligned} \beta_{0j} &= \gamma_{00} + \gamma_{01} W_j + u_{0j} \\ \beta_{1j} &= \gamma_{10} + \gamma_{11} W_j + u_{1j} \end{aligned}}$$
- One then has two set of βs estimates.
- HLM provides an overall weighted average estimate of the using these two sets of β estimates. The weights are obtained by partitioning the observed variances of βs into a true variance vs. error variance components. The resulting composite β estimates produces a smaller mean square error term than either the Level 1- and Level 2-estimates.

Estimation

- The first estimator of the βs is simply the OLS regression estimator based on data from team j: $OLS(\hat{\beta}_j) = (X_j^{'}X_j)^{-1}X_j^{'}Y_j$
- The second estimator is the predicted value of β_j given team culture captured by W_j: $Second(\hat{\beta}_j) = W_j\hat{\gamma}$
- The optimal combination of these two estimators is:

$$optimal(\beta_j) = \Lambda_j\hat{\beta}_j + (I - \Lambda_j)W_j\hat{\gamma}$$

For each level of Level 2:

$$\lambda_{0j} = \tau_{00} / (\tau_{00} + \nu_{0j})$$

$$\lambda_{1j} = \tau_{11} / (\tau_{11} + \nu_{1j})$$

Where $\Lambda_j = T\Delta_j^{-1} = T(T + V_j)^{-1}$

$$\nu_{0j} = \frac{\sigma^2}{n_j} \quad \nu_{1j} = \frac{\sigma^2}{\sum x_{ij}^2}$$

T is the parameter dispersion (variance of the βs or the τ matrix)

V_j is the error dispersion (variance of random error within each group)

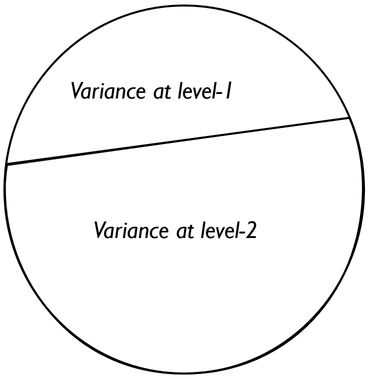
Λ_j is the ratio of true variance to total variance or the **reliability** of the parameters

Percentage of variance accounted for

$$y_{ij} = \beta_{0j} + \beta_{1j}(x_{ij} - \bar{x}_{.j}) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01}w_j + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11}w_j + u_{1j}$$



- How much of the variances of y_{ij} is due to level-1 and how much to level-2?
- How much variance of y_{ij} is accounted for at level-1, how much as level-2?

Percentage of variance explained when there is a group-level predictor

$$\% \text{ varaince explained in } \beta_{qj} = \frac{\hat{\tau}_{qq}(\text{random regression}) - \hat{\tau}_{qq}(\text{fitted model})}{\hat{\tau}_{qq}(\text{random regression})}$$

Random Regression Model

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + u_{1j}$$

$\hat{\tau}_{qq}(\text{random regression})$

Fitted Model

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11} W_j + u_{1j}$$

$\hat{\tau}_{qq}(\text{fitted model})$

Percentage of variance explained in HLM

Model 1

$$y_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$

Model 2

$$y_{ij} = \beta_{0j} + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01}(MEANSES)_j + u_{0j}$$

Model 3

$$y_{ij} = \beta_{0j} + \beta_{1j}(SES_{ij} - MEANSES_{.j}) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{0j} = \gamma_{10} + u_{1j}$$

Model 4

$$y_{ij} = \beta_{0j} + \beta_{1j}(SES_{ij} - MEANSES_{.j}) + r_{ij}$$
$$\beta_{0j} = \gamma_{00} + \gamma_{01}(MEANSES)_j + u_{0j}$$
$$\beta_{0j} = \gamma_{10} + \gamma_{11}(MEANSES)_j + u_{1j}$$

	Var	Model 1	Model 2	Model 3	Model 4
r_{ij}	σ^2	39.15	39.16	36.71	36.71
u_{0j}	τ_{00}	8.62	2.64	8.68	2.65
u_{1j}	τ_{11}			.68	.66

% variance accounted for

	Var	Model 1	Model 2	Model 3	Model 4
		baseline	Level2 predictor	Level1 predictor	Level1&2 predictor
r_{ij}	σ^2	39.15	39.16	36.71	36.71
u_{0j}	τ_{00}	8.62	2.64	8.68	2.65
u_{1j}	τ_{11}			.68	.66

$$Ratio\ 1 = \frac{\tau_{00}(M_1)}{\tau_{00}(M_1) + \sigma^2(M_1)} = \frac{8.62}{8.62 + 39.15} = .1804$$

18% of variance of Math attributable to level 2 (school level);
82% of variance is at the individual level (level 1).

% variance accounted for: Level 1

	Var	Model 1	Model 2	Model 3	Model 4
		baseline	Level2 predictor	Level1 predictor	Level1&2 predictor
r_{ij}	σ^2	39.15	39.16	36.71	36.71
u_{0j}	τ_{00}	8.62	2.64	8.68	2.65
u_{1j}	τ_{11}			.68	.66

$$Ratio\ 2 = \frac{\sigma^2(M_1) - \sigma^2(M_3)}{\sigma^2(M_1)} = \frac{39.15 - 36.71}{39.15} = .0623$$

6% of individual Math variance is explained by SES.

% variance accounted for: level 2

	Var	Model 1	Model 2	Model 3	Model 4
		baseline	Level2 predictor	Level1 predictor	Level1&2 predictor
r_{ij}	σ^2	39.15	39.16	36.71	36.71
u_{0j}	τ_{00}	8.62	2.64	8.68	2.65
u_{1j}	τ_{11}			.68	.66

$$Ratio\ 3 = \frac{\tau_{00}(M_1) - \tau_{00}(M_2)}{\tau_{00}(M_1)} = \frac{8.62 - 2.64}{8.62} = .6937$$

The level 1 variable SES helped tremendously (↓69.37%) to explain the math performance across schools.

% variance accounted for: intercept/slope

	Var	Model 1	Model 2	Model 3	Model 4
		baseline	Level2 predictor	Level1 predictor	Level1&2 predictor
r_{ij}	σ^2	39.15	39.16	36.71	36.71
u_{0j}	τ_{00}	8.62	2.64	8.68	2.65
u_{1j}	τ_{11}			.68	.66

$$Ratio\ 4a = \frac{\tau_{00}(M_3) - \tau_{00}(M_4)}{\tau_{00}(M_3)} = \frac{8.68 - 2.65}{8.68} = .6947$$

$$Ratio\ 4b = \frac{\tau_{11}(M_3) - \tau_{11}(M_4)}{\tau_{11}(M_3)} = \frac{.68 - .66}{.68} = .0002$$

Adding MEANSES explained intercept, but not slope.

% variance accounted for

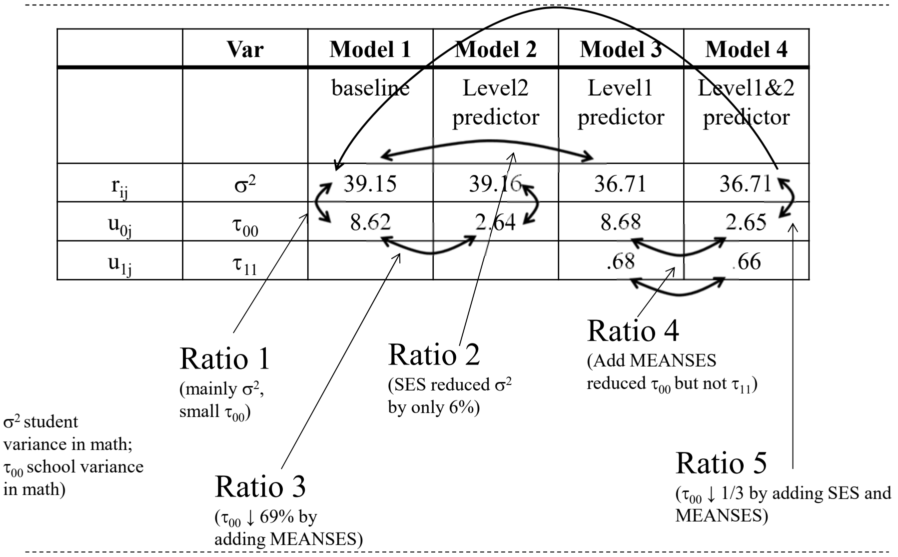
	Var	Model 1	Model 2	Model 3	Model 4
		baseline	Level2 predictor	Level1 predictor	Level1&2 predictor
r_{ij}	σ^2	39.15	39.16	36.71	36.71
u_{0j}	τ_{00}	8.62	2.64	8.68	2.65
u_{1j}	τ_{11}	18%		.68	.66

6.7%

$$Ratio\ 5 = \frac{\tau_{00}(M_4)}{\tau_{00}(M_4)+\sigma^2(M_4)} = \frac{2.65}{2.65+36.71} = .0673$$

When compare M_1 and M_4, the % level 2 residual variance τ_{00} dropped from 18.04% of total residual variance to 6.73% (about 1/3), meaning that MEANSES is a more effective predictor than individual SES.

% variance accounted for



	Var	Model 1	Model 2	Model 3	Model 4
		baseline	Level2 predictor	Level1 predictor	Level1&2 predictor
r_{ij}	σ^2	39.15	39.16	36.71	36.71
u_{0j}	τ_{00}	8.62	2.64	8.68	2.65
u_{1j}	τ_{11}			.68	.66

Ratio 1
(mainly σ^2, small τ_{00})

Ratio 2
(SES reduced σ^2 by only 6%)

Ratio 4
(Add MEANSES reduced τ_{00} but not τ_{11})

σ^2 student variance in math; τ_{00} school variance in math)

Ratio 3
(τ_{00} ↓ 69% by adding MEANSES)

Ratio 5
(τ_{00} ↓ 1/3 by adding SES and MEANSES)

Multiple Xs

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{1ij} - \overline{X}_{1.j}\right) + \beta_{2j}\left(X_{2ij} - \overline{X}_{2.j}\right) + r_{ij}$$

As discussed β_{0j} and β_{1j} are modeled as left:

$$\boxed{\begin{aligned}\beta_{0j} &= \gamma_{00} + \gamma_{01}W_j + u_{0j}\\ \beta_{1j} &= \gamma_{10} + \gamma_{11}W_j + u_{1j}\end{aligned}}$$

β_{2j} can be modeled in four different forms:

1. The effect of X_{2ij} is constrained to be invariant across Level-2 units;

   $\boldsymbol{\beta_{2j} = \gamma_{20}}$

2. The slope β_{2j} is a function of an average value γ_{20}, plus a random effect associated with each Level-2 unit;

   $\boldsymbol{\beta_{2j} = \gamma_{20} + u_{2j}}$

3. Part of the variation of the slop β_{2j} can be predicted by W_j, but a random components, u_{2j}, remains unexplained;

   $\boldsymbol{\beta_{2j} = \gamma_{20} + \gamma_{21} W_j + u_{2j}}$

4. Once the effect of W_j is taken into account, the residual variation in β_{2j} [Var(u_{2j})=τ_{22}] is negligible.

   $\boldsymbol{\beta_{2j} = \gamma_{20} + \gamma_{21} W_j}$

Level 1 & 2 models

$$\boxed{Y_{ij} = \beta_{0j} + \beta_{1j}\left(X_{ij} - \overline{X}_{.j}\right) + r_{ij}}$$

$$\boxed{\begin{aligned}\beta_{0j} &= \gamma_{00} + \gamma_{01}W_j + u_{0j}\\ \beta_{1j} &= \gamma_{10} + \gamma_{11}W_j + u_{1j}\end{aligned}}$$

Level	1	2
Units	subordinates	teams
Errors (random effects)	r_{ij}	u_{0j} , u_{1j}
Variance	Var(r_{ij}) $= \sigma^2$	Var(u_{0j}) , Var(u_{1j}) , Cov(u_{0j} , u_{1j})
Parameters	β	γ
Predictor	X_{ij}	W_j

The End